Precalculus

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Preface

There are very few good Calculus books, written in English, available to the American reader. Only [Har], [Kla], [Apo], [Olm], and [Spi] come to mind.

The situation in Precalculus is even worse, perhaps because Precalculus is a peculiar American animal: it is a review course of all that which should have been learned in High School but was not. A distinctive American slang is thus called to describe the situation with available Precalculus textbooks: they stink!

I have decided to write these notes with the purpose to, at least locally, for my own students, I could ameliorate this situation and provide a semi-rigorous introduction to precalculus.

I try to follow a more or less historical approach. My goal is to not only present a coherent view of Precalculus, but also to instill appreciation for some elementary results from Precalculus. Thus I do not consider a student (or for that matter, *an instructor*) to be educated in Precalculus if he cannot demonstrate that $\sqrt{2}$ is irrational;[1] that the equation of a non-vertical line on the plane is of the form $y = mx + k$, and conversely; that lines $y = m_1x + k_1$ and $y = m_2x + k_2$ are perpendicular if and only if $m_1m_2 = -1$; that the curve with equation $y = x^2$ is a parabola, etc.

I do not claim a 100% rate of success, or that I stick to the same paradigms each semester,[2] but a great number of students seem genuinely appreciative what I am trying to do.

I start with sets of real numbers, in particular, intervals. I try to make patent the distinction between rational and irrational numbers, and their decimal representations. Usually the students reaching this level have been told fairy tales about $\sqrt{2}$ and π being irrational. I prove the irrationality of the former using Hipassus of Metapontum's proof.[3]

After sets on the line, I concentrate on distance on the line. Absolute values are a good place (in my opinion) to introduce sign diagrams, which are a technique that will be exploited in other instances, as for example, in solving rational and absolute-value inequalities.

The above programme is then raised to the plane. I derive the distance formula from the Pythagorean Theorem. It is crucial, in my opinion, to make the students understand that these formulæ do not appear by fiat, but that are obtained from previous concepts.

Depending on my mood, I either move to the definition of functions, or I continue to various curves. Let us say for the sake of argument that I have chosen to continue with curves.

Once the distance formula is derived, it is trivial to talk about circles and semi-circles. The graph of $y = \sqrt{1-x^2}$ is obtained. This is the first instance of the translation *Geometry-to-Algebra* and *Algebra-to-Geometry* that the students see, that is, they are able to tell what the equation of a given circle looks like, and vice-versa, to produce a circle from an equation.

Now, using similar triangles and the distance formula once again, I move on to lines, proving that the canonical equation of a non-vertical line is of the form $y = mx + k$ and conversely. I also talk about parallel and normal lines, proving[4] that two non-vertical lines are perpendicular if and only if the product of their slopes is -1. In particular, the graph of $y = x$, $y = -x$, and $y = |x|$ are obtained.

The next curve we study is the parabola. First, I give the locus definition of a parabola. We use a T-square and a string in order to illustrate the curve produced by the locus definition. It turns out to be a sort-of "U"-shaped curve. Then, using the distance formula again, we prove that one special case of these parabolas has equation $y = x^2$. The graph of $x = y^2$ is obtained, and from this the graph of $y = \sqrt{x}$.

Generally, after all this I give my first exam.

We now start with functions. A *function* is defined by means of the following five characteristics:

[1] Plato's dictum comes to mind: "He does not deserve the appellative *man* who does not know that the diagonal of a square is inconmensurable with its side.

[2] I don't, in fact, I try to change emphases from year to year.

[3] I wonder how many of my colleagues know how to prove that π is irrational? Transcendental? Same for e, $\log 2$, $\cos 1$, etc. How many tales are the students told for which the instructor does not know the proof?

[4] The Pythagorean Theorem once again!

1. a set of inputs, called the *domain* of the function;
2. a set of all *possible* outputs, called the *target set* of the function;
3. a name for a typical input (colloquially referred to as the *dummy variable*);
4. a name for the function;
5. an assignment rule or formula that assigns to every element of the domain a unique element of the target set.

All these features are collapsed into the notation

$$f: \begin{array}{ccc} \textbf{Dom}(f) & \to & \textbf{Target}(f) \\ x & \mapsto & f(x) \end{array}.$$

Defining functions in such a careful manner is necessary. Most American books focus only on the assignment rule (formula), but this makes a mess later on in abstract algebra, linear algebra, computer programming etc. For example, even though the following four functions have the same formula, they are all different:

$$a: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}; \qquad b: \begin{array}{ccc} [0;+\infty[& \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array};$$

$$c: \begin{array}{ccc} \mathbb{R} & \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}; \qquad d: \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array};$$

for a is neither injective nor surjective, b is injective but not surjective, c is surjective but not injective, and d is a bijection.

I first focus on the domain of the function. We study which possible sets of real numbers can be allowed so that the output be a real number.

I then continue to graphs of functions and functions defined by graphs.[5] At this point, of course, there are very functional curves of which the students know the graphs: only $x \mapsto x$, $x \mapsto |x|$, $x \mapsto x^2$, $x \mapsto \sqrt{x}$, $x \mapsto \sqrt{1-x^2}$, piecewise combinations of them, etc., but they certainly can graph a function with a finite (and extremely small domain). The repertoire is then extended by considering the following transformations of a function f: $x \mapsto -f(x)$, $x \mapsto f(-x)$, $x \mapsto Vf(Hx+h)+v$, $x \mapsto |f(x)|$, $x \mapsto f(|x|)$, $x \mapsto f(-|x|)$. These last two transformations lead a discussion about even and odd functions. The floor, ceiling, and the decimal part functions are also now introduced.

The focus now turns to the assignment rule of the function, and is here where the algebra of functions (sum, difference, product, quotient, composition) is presented. Students are taught the relationship between the various domains of the given functions and the domains of the new functions obtained by the operations.

Composition leads to iteration, and iteration leads to inverse functions. The student now becomes familiar with the concepts of injective, surjective, and bijective functions. The relationship between the graphs of a function and its inverse are explored. It is now time for the second exam.

The distance formula is now powerless to produce the graph of more complicated functions. The concepts of *monotonicity* and *convexity* of a function are now introduced. Power functions (with strictly positive integral exponents are now studied. The global and local behaviour of them is studied, obtaining a catalogue of curves $y = x^n$, $n \in \mathbb{N}$.

After studying power functions, we now study polynomials. The study is strictly limited to polynomials whose splitting field is $\mathbb{R}$.[6]

We now study power functions whose exponent is a strictly negative integer. In particular, the graph of the curve $xy = 1$ is deduced from the locus definition of the hyperbola. Studying the monotonicity and concavity of these functions, we obtain a catalogue of curves $y = x^{-n}$, $n \in \mathbb{N}$.

[5] This last means, given a picture in $\mathbb{R}^2$ that passes the vertical line test, we derive its domain and image by looking at its shadow on the x and y axes.

[6] I used to make a brief incursion into some ancillary topics of the theory of equations, but this makes me digress too much from my plan of *Algebra-Geometry-Geometry-Algebra*, and nowadays I am avoiding it. I have heard colleagues argue for Ruffini's Theorem, solely to be used in one example of Calculus I, the factorisation of a cubic or quartic polynomial in optimisation problems, but it seems hardly worth the deviation for only such an example.

Rational functions are now introduced, but only those whose numerators and denominators are polynomials splitting in $\mathbb{R}$. The problem of graphing them is reduced to examining the local at the zeroes and poles, and their global behaviour.

I now introduce formulæ of the type $x \mapsto x^{1/n}$, $n \in \mathbb{Z} \setminus \{0\}$, whose graphs I derived by means of inverse functions of $x \mapsto x^n$, $n \in \mathbb{Z}$. This concludes the story of Precalculus I as I envision it, and it is time for the third exam, usually during the last week of classes. A comprehensive final exam is given during final-exam week.

These notes are in constant state of revision. I would greatly appreciate comments, additions, exercises, figures, etc., in order to help me enhance them.

David A. Santos

To the Student

These notes are provided for your benefit as an attempt to organise the salient points of the course. They are a *very terse* account of the main ideas of the course, and are to be used mostly to refer to central definitions and theorems. The number of examples is minimal. The *motivation* or informal ideas of looking at a certain topic, the ideas linking a topic with another, the worked-out examples, etc., are given in class. Hence these notes are not a substitute to lectures: **you must always attend to lectures**. The order of the notes may not necessarily be the order followed in the class.

There is a certain algebraic fluency that is necessary for a course at this level. These algebraic prerequisites would be difficult to codify here, as they vary depending on class response and the topic lectured. If at any stage you stumble in Algebra, seek help! I am here to help you!

Tutoring can sometimes help, but bear in mind that whoever tutors you may not be familiar with my conventions. Again, I am here to help! On the same vein, other books may help, but the approach presented here is at times unorthodox and finding alternative sources might be difficult.

Here are more recommendations:

- Read a section before class discussion, in particular, read the definitions.
- Class provides the informal discussion, and you will profit from the comments of your classmates, as well as gain confidence by providing your insights and interpretations of a topic. **Don't be absent!**
- I encourage you to form study groups and to discuss the assignments. Discuss among yourselves and help each other but don't be *parasites!* Plagiarising your classmates' answers will only lead you to disaster!
- Once the lecture of a particular topic has been given, take a fresh look at the notes of the lecture topic.
- Try to understand a single example well, rather than ill-digest multiple examples.
- Start working on the distributed homework ahead of time.
- **Ask questions during the lecture.** There are two main types of questions that you are likely to ask.
    1. *Questions of Correction: Is that a minus sign there?* If you think that, for example, I have missed out a minus sign or wrote P where it should have been Q,[7] then by all means, ask. No one likes to carry an error till line XLV because the audience failed to point out an error on line I. Don't wait till the end of the class to point out an error. Do it when there is still time to correct it!
    2. *Questions of Understanding: I don't get it!* Admitting that you do not understand something is an act requiring utmost courage. But if you don't, it is likely that many others in the audience also don't. On the same vein, if you feel you can explain a point to an inquiring classmate, I will allow you time in the lecture to do so. The best way to ask a question is something like: "How did you get from the second step to the third step?" or "What does it mean to complete the square?" Asseverations like "I don't understand" do not help me answer your queries. If I consider that you are asking the same questions too many times, it may be that you need extra help, in which case we will settle what to do outside the lecture.
- Don't fall behind! The sequence of topics is closely interrelated, with one topic leading to another.
- You will need square-grid paper, a ruler (preferably a T-square), some needle thread, and a compass.
- The use of calculators is allowed, especially in the occasional lengthy calculations. However, when graphing, you will need to provide algebraic/analytic/geometric support of your arguments. The questions on assignments and exams will be posed in such a way that it will be of no advantage to have a graphing calculator.
- Presentation is critical. Clearly outline your ideas. When writing solutions, outline major steps and write in complete sentences. As a guide, you may try to emulate the style presented in the scant examples furnished in these notes.

[7] My doctoral adviser used to say "I said A, I wrote B, I meant C and it should have been D!

Notation

$\in$	Belongs to.
$\notin$	Does not belong to.
$\forall$	For all (Universal Quantifier).
$\exists$	There exists (Existential Quantifier).
$\varnothing$	Empty set.
$P \implies Q$	P implies Q.
$P \Leftrightarrow Q$	P if and only if Q.
$\mathbb{N}$	The Natural Numbers $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{Z}$	The Integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
$\mathbb{Q}$	The Rational Numbers.
$\mathbb{R}$	The Real Numbers.
$\mathbb{C}$	The Complex Numbers.
A^n	The set of n-tuples $\{(a_1, a_2, \ldots, a_n) \vert a_k \in A\}$.
$]a; b[$	The open finite interval $\{x \in \mathbb{R} : a < x < b\}$.
$[a; b]$	The closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$.
$]a; b]$	The semi-open interval $\{x \in \mathbb{R} : a < x \leq b\}$.
$[a; b[$	The semi-closed interval $\{x \in \mathbb{R} : a \leq x < b\}$.
$]a; +\infty[$	The infinite open interval $\{x \in \mathbb{R} : x > a\}$.
$]-\infty; a]$	The infinite closed interval $\{x \in \mathbb{R} : x \leq a\}$.
$\sum_{k=1}^{n} a_k$	The sum $a_1 + a_2 + \cdots + a_{n-1} + a_n$.

1 The Line

This chapter introduces essential notation and terminology that will be used throughout these notes. The focus of this course will be the real numbers, of which we assume the reader has passing familiarity. We will review some of the properties of real numbers as a way of having a handy vocabulary that will be used for future reference.

1.1 Sets and Notation

1 Definition We will mean by a *set* a collection of well defined members or *elements*. A *subset* is a sub-collection of a set. We denote that B is a subset of A by the notation $B \subseteqq A$ or sometimes $B \subset A$.[1]

Some sets of numbers will be referred to so often that they warrant special notation. Here are some of the most common ones.

$\varnothing$	Empty set.
$\mathbb{N}$	The Natural Numbers $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{Z}$	The Integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
$\mathbb{Q}$	The Rational Numbers.
$\mathbb{R}$	The Real Numbers.
$\mathbb{C}$	The Complex Numbers.

Observe that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

From time to time we will also use the following notation, borrowed from set theory and logic.

$\in$	Is in. Belongs to. Is an element of.
$\notin$	Is not in. Does not belong to. Is not an element of.
$\forall$	For all (Universal Quantifier).
$\exists$	There exists (Existential Quantifier).
$P \implies Q$	P implies Q.
$P \Leftrightarrow Q$	P if and only if Q.

2 Example $-1 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$.

3 Definition Let A be a set. If a belongs to the set A, then we write $a \in A$, read "a is an element of A." If a does not belong to the set A, we write $a \notin A$, read "a is not an element of A." The set that has no elements, that is *empty set*, will be denoted by $\varnothing$.

There are various ways of alluding to a set. We may use a description, or we may list its elements individually.

4 Example The sets

$$A = \{x \in \mathbb{Z} : x^2 \leq 9\}, \qquad B = \{x \in \mathbb{Z} : |x| \leq 3\}, \qquad C = \{-3, -2, -1, 0, 1, 2, 3\}$$

are identical. The first set is the set of all integers whose square lies between 1 and 9 inclusive, which is precisely the second set, which again is the third set.

5 Example Consider the set

$$A = \{2, 9, 16, \ldots, 716\},$$

where the elements are in arithmetic progression. How many elements does it have? Is $401 \in A$? Is $514 \in A$? What is the sum of the elements of A?

[1] There is no agreement relating the choice. Some use $\subset$ to denote strict containment, that is, $A \subseteqq B$ but $A \neq B$. In the case when we want to denote strict containment we will simply write $A \subsetneqq B$.

Solution: ► *Observe that the elements have the form*

$$2 = 2+7\cdot 0, \quad 9 = 2+7\cdot 1, \quad 16 = 2+7\cdot 2, \quad \ldots,$$

thus the general element term has the form $2+7n$*. Now,*

$$2+7n = 716 \implies n = 102.$$

This means that there are 103 *elements, since we started with* $n = 0$*.*

If $2+7k = 401$*, then* $k = 57$*, so* $401 \in A$*. On the other hand,* $2+7a = 514 \implies a = \dfrac{512}{7}$*, which is not integral, and hence* $514 \notin A$*.*

To find the sum of the arithmetic progression we will use a trick due to the great German mathematician K. F. Gauß who presumably discovered it when he was in first grade. To add the elements of A, put

$$S = 2+9+16+\cdots+716.$$

Observe that the sum does not change if we sum it backwards, so

$$S = 716+709+702+\cdots+16+9+2.$$

Adding both sums and grouping corresponding terms,

$$\begin{aligned} 2S &= (2+716)+(9+709)+(16+702)+\cdots+(702+16)+(709+9)+(716+2) \\ &= 718+718+718+\cdots+718+718+718 \\ &= 718\cdot 103, \end{aligned}$$

since there are 103 *terms. We deduce that*

$$S = \frac{718\cdot 103}{2} = 36977.$$

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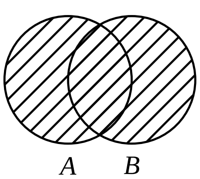

Figure 1.1: $A \cup B$

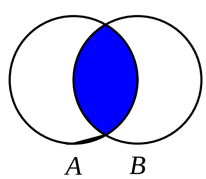

Figure 1.2: $A \cap B$

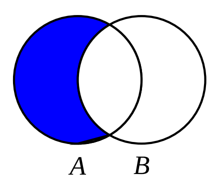

Figure 1.3: $A \setminus B$

We now define some operations with sets.

6 Definition The *union* of two sets A and B, is the set

$$A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}.$$

This is read "A union B." See figure 1.1.

The *intersection* of two sets A and B, is

$$A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}.$$

Interval Notation	**Set Notation**	**Graphical Representation**
$[a;b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$[2]	
$]a;b[$	$\{x \in \mathbb{R} : a < x < b\}$	
$[a;b[$	$\{x \in \mathbb{R} : a \leq x < b\}$	
$]a;b]$	$\{x \in \mathbb{R} : a < x \leq b\}$	
$]a;+\infty[$	$\{x \in \mathbb{R} : x > a\}$	
$[a;+\infty[$	$\{x \in \mathbb{R} : x \geq a\}$	
$]-\infty;b[$	$\{x \in \mathbb{R} : x < b\}$	
$]-\infty;b]$	$\{x \in \mathbb{R} : x \leq b\}$	
$]-\infty;+\infty[$	$\mathbb{R}$	

Table 1.1: Intervals.

This is read "A intersection B." See figure 1.2.

The *difference* of two sets A and B, is

$$A \setminus B = \{x : (x \in A) \text{ and } (x \notin B)\}.$$

This is read "A set minus B." See figure 1.3.

7 Example Let $A = \{1,2,3,4,5,6\}$, and $B = \{1,3,5,7,9\}$. Then

$$A \cup B = \{1,2,3,4,5,6,7,9\}, \quad A \cap B = \{1,3,5\}, \quad A \setminus B = \{2,4,6\}, \quad B \setminus A = \{7,9\}.$$

8 Example Consider the sets of arithmetic progressions

$$A = \{3,9,15,\ldots,681\}, \qquad B = \{9,14,19,\ldots,564\}.$$

How many elements do they share, that is, how many elements does $A \cap B$ have?

Solution: ► *The members of A have common difference 6 and the members of B have common difference 5. Since the least common multiple of 6 and 5 is 30, and 9 is the smallest element that A and B have in common, every element in $A \cap B$ has the form $9+30k$. We then need the largest $k \in \mathbb{N}$ satisfying the inequality*

$$9 + 30k \leq 564 \implies k \leq 18.5,$$

and since k is integral, the largest value it can achieve is 18. Thus $A \cap B$ has $18+1 = 19$ elements, where we have added 1 because we start with $k = 0$. In fact,

$$A \cap B = \{9,39,69,\ldots,549\}.$$

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9 Definition An *interval* I is a subset of the real numbers with the following property: if $s \in I$ and $t \in I$, and if $s < x < t$, then $x \in I$. In other words, intervals are those subsets of real numbers with the property that every number between two elements is also contained in the set. Since there are infinitely many decimals between two different real numbers, intervals with distinct endpoints contain infinitely many members. Table 1.1 shews the various types of intervals.

Observe that we indicate that the endpoints are included by means of shading the dots at the endpoints and that the endpoints are excluded by not shading the dots at the endpoints. [3]

10 Example If $A = [-10;2]$, $B =]-\infty;1[$, then

$$A \cap B = [-10;1[, \quad A \cup B =]-\infty;2], \quad A \setminus B = [1;2], \quad B \setminus A =]-\infty;-10[.$$

11 Example Let $A = \left[1-\sqrt{3};1+\sqrt{2}\right]$, $B = \left[\frac{\pi}{2};\pi\right[$. By approximating the endpoints to three decimal places, we find $1-\sqrt{3} \approx -0.732$, $1+\sqrt{2} \approx 2.414$, $\frac{\pi}{2} \approx 1.571$, $\pi \approx 3.142$. Thus

$$A \cap B = \left[\frac{\pi}{2};1+\sqrt{2}\right], \quad A \cup B = \left[1-\sqrt{3};\pi\right[, \quad A \setminus B = \left[1-\sqrt{3};\frac{\pi}{2}\right[, \quad B \setminus A = \left]1+\sqrt{2};\pi\right[.$$

We conclude this section by defining some terms for future reference.

12 Definition Let $a \in \mathbb{R}$. We say that the set $\mathscr{N}_a \subseteqq \mathbb{R}$ is a *neighbourhood* of a if there exists an open interval I centred at a such that $I \subseteqq \mathscr{N}_a$. In other words, $\mathscr{N}_a$ is a neighbourhood of a if there exists a $\delta > 0$ such that $]a-\delta;a+\delta[\subseteqq \mathscr{N}_a$. This last condition may be written in the form

$$\{x \in \mathbb{R} : |x-a| < \delta\} \subseteqq \mathscr{N}_a.$$

If $\mathscr{N}_a$ is a neighbourhood of a, then we say that $\mathscr{N}_a \setminus \{a\}$ is a *deleted neighbourhood of* a.

This means that $\mathscr{N}_a$ is a neighbourhood of a if a has neighbours left and right.

13 Example The interval $]0;1[$ is neighbourhood of all of its points. The interval $[0;1]$, on the contrary, is a neighbourhood of all of its points, with the exception of its endpoints 0 and 1, since 0 does not have left neighbours in the interval and 1 does not have right neighbours on the interval.

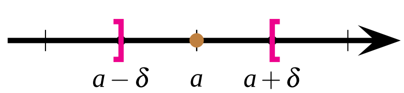

Figure 1.4: Neighbourhood of a.

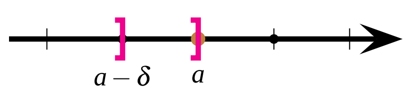

Figure 1.5: Sinistral neighbourhood of a.

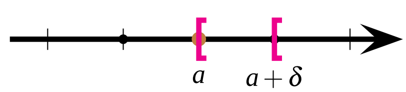

Figure 1.6: Dextral neighbourhood of a.

We may now extend the definition of neighbourhood.

14 Definition Let $a \in \mathbb{R}$. We say that the set $V \subseteqq \mathbb{R}$ is a *dextral neighbourhood* or *right-hand neighbourhood* of a if there exists a $\delta > 0$ such that $[a;a+\delta[\subseteqq V$. We say that the set $V' \subseteqq \mathbb{R}$ is a *sinistral neighbourhood* or *left-hand neighbourhood* of a if there exists a $\delta' > 0$ such that $]a-\delta';a] \subseteqq V'$.

The following result will be used later.

[3] It may seem like a silly analogy, but think that in $[a;b]$ the brackets are "arms" "hugging" a and b, but in $]a;b[$ the "arms" are repulsed. "Hugging" is thus equivalent to including the endpoint, and "repulsing" is equivalent to excluding the endpoint.

15 Lemma Let $(a,b) \in \mathbb{R}^2, a < b$. Then every number of the form $\lambda a + (1-\lambda)b$, $\lambda \in [0;1]$ belongs to the interval $[a;b]$. Conversely, if $x \in [a;b]$ then we can find a $\lambda \in [0;1]$ such that $x = \lambda a + (1-\lambda)b$.

Proof: *Clearly* $\lambda a + (1-\lambda)b = b + \lambda(a-b)$ *and since* $a - b < 0$,

$$b = b + 0(a-b) \geq b + \lambda(a-b) \geq b + 1(a-b) = a,$$

whence the first assertion follows.

Assume now that $x \in [a;b]$. *Solve the equation* $x = \lambda a + (1-\lambda)b$ *for* λ *obtaining* $\lambda = \frac{x-b}{b-a}$. *All what remains to prove is that* $0 \leq \lambda \leq 1$, *but this is evident, as* $0 \leq x - b \leq b - a$. *This concludes the proof.* ❑

Homework

1.1.1 Problem List all the elements of the set

$$\{x \in \mathbb{Z} : 1 \leq x^2 \leq 100, \quad x \text{ is divisible by } 3\}.$$

1.1.2 Problem Determine the set

$$\{x \in \mathbb{N} : x^2 - x = 6\}$$

explicitly.

1.1.3 Problem Determine the set of numerators of all the fractions lying strictly between 2 and 3 that have denominator 6, that is, determine the set

$$\{x \in \mathbb{N} : 2 < \frac{x}{6} < 3\}$$

explicitly.

1.1.4 Problem Let $A = \{a,b,c,d,e,f\}$ and $B = \{a,e,i,o,u\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$.

1.1.5 Problem Describe the following sets explicitly by either providing a list of their elements or an interval.

1. $\{x \in \mathbb{R} : x^3 = 8\}$
2. $\{x \in \mathbb{R} : |x|^3 = 8\}$
3. $\{x \in \mathbb{R} : |x| = -8\}$
4. $\{x \in \mathbb{R} : |x| < 4\}$
5. $\{x \in \mathbb{Z} : |x| < 4\}$
6. $\{x \in \mathbb{R} : |x| < 1\}$
7. $\{x \in \mathbb{Z} : |x| < 1\}$
8. $\{x \in \mathbb{Z} : x^{2002} < 0\}$

1.1.6 Problem Describe explicitly the set

$$\{x \in \mathbb{Z} : x < 0, 1000 < x^2 < 2003\}$$

by listing its elements.

1.1.7 Problem The set S is formed according to the following rules:

1. 2 belongs to S;
2. if n is in S then $n+5$ is also in S;
3. if n is in S then $3n$ is also in S.

Find the largest integer in the set

$$\{1, 2, 3, \ldots, 2008\}$$

that does not belong to S.

1.1.8 Problem Use the trick of Gauß to prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

1.1.9 Problem Let $C =]-5;5[$, $D =]-1;+\infty[$. Find $C \cap D$, $C \cup D$, $C \setminus D$, and $D \setminus C$.

1.1.10 Problem Let $C =]-5;3[$, $D = [4;+\infty[$. Find $C \cap D$, $C \cup D$, $C \setminus D$, and $D \setminus C$.

1.1.11 Problem Let $C = \left[-1;-2+\sqrt{3}\right[$, $D = \left[-0.5;\sqrt{2}-1\right]$. Find $C \cap D$, $C \cup D$, $C \setminus D$, and $D \setminus C$.

1.1.12 Problem Consider 101 different points $x_1, x_2, \ldots, x_{101}$ belonging to the interval $[0;1[$. Shew that there are at least two say x_i and $x_j, i \neq j$, such that

$$|x_i - x_j| \leq \frac{1}{100}$$

1.1.13 Problem (Dirichlet's Approximation Theorem) Shew that $\forall x \in \mathbb{R}$, $\forall N \in \mathbb{N}, N > 1$, $\exists (h \in \mathbb{N}, k \in \mathbb{N})$ with $0 < k \leq N$ such that

$$\left| x - \frac{h}{k} \right| < \frac{1}{Nk}.$$

1.2 Rational Numbers and Irrational Numbers

Let us start by considering the strictly positive natural numbers. Primitive societies needed to count objects, say, their cows or sheep. Though some societies, like the Yanomame indians in Brazil or members of the CCP English and Social Sciences Department[4] cannot count above 3, the need for counting is indisputable. In fact, many of these societies were able to make the

[4] Among these, many are Philosophers, who, though unsuccessful in finding their Philosopher's Stone, have found renal calculi.

following abstraction: add to a pile one pebble (or stone) for every sheep, in other words, they were able to make one-to-one correspondences. In fact, the word *Calculus* comes from the Latin for "stone."

Breaking an object into almost equal parts (that is, *fractioning* it) justifies the creation of the positive rational numbers. In fact, most ancient societies did very well with just the strictly positive rational numbers. The problems of counting and of counting broken pieces were solved completely with these numbers.

As societies became more and more sophisticated, the need for new numbers arose. For example, it is believed that the introduction of negative quantities arose as an accounting problem in Ancient India. Fair enough, write $+1$ if you have a rupee—or whatever unit that ancient accountant used—in your favour. Write -1 if you owe one rupee. Write 0 if you are rupeeless.

Thus we have constructed $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. In $\mathbb{Q}$ we have, so far, a very elegant system of numbers which allows us to perform four arithmetic operations (addition, subtraction, multiplication, and division)[5]and that has the notion of "order", which we will discuss in a latter section. A formal definition of the rational numbers is the following.

16 Definition The set of rational numbers $\mathbb{Q}$ is the set of quotients of integers where a denominator 0 is not allowed. In other words:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Notice also that $\mathbb{Q}$ has the wonderful property of *closure*, meaning that if we add, subtract, multiply or divide any two rational numbers (with the exclusion of division by 0), we obtain as a result a rational number, that is, we stay within the same set.

Since $a = \frac{a}{1}$, every integer is also a rational number, in other words, $\mathbb{Z} \subseteq \mathbb{Q}$. Notice that every finite decimal can be written as a fraction, for example, we can write the decimal 3.14 as

$$3.14 = \frac{314}{100} = \frac{157}{50}.$$

What about non-finite decimals? Can we write them as a fraction? The next example shews how to convert an infinitely repeating decimal to fraction from.

17 Example Write the infinitely repeating decimal $0.3\overline{45} = 0.345454545\ldots$ as the quotient of two natural numbers.

Solution: ▸ *The trick is to obtain multiples of $x = 0.345454545\ldots$ so that they have the same infinite tail, and then subtract these tails, cancelling them out.*[6] *So observe that*

$$10x = 3.45454545\ldots; 1000x = 345.454545\ldots \implies 1000x - 10x = 342 \implies x = \frac{342}{990} = \frac{19}{55}.$$

◂

By mimicking the above examples, the following should be clear: decimals whose decimal expansions terminate or repeat are rational numbers. Since we are too cowardly to prove the next statement,[7] we prefer to call it a

18 Fact Every rational number has a terminating or a repeating decimal expansion. Conversely, a real number with a terminating or repeating decimal expansion must be a rational number. Moreover, a rational number has a terminating decimal expansion if and only if its denominator is of the form 2^m5^n, where m and n are natural numbers.

From the above fact we can tell, without actually carrying out the long division, that say, $\frac{1}{1024} = \frac{1}{2^{10}}$ has a terminating decimal expansion, but that, say, $\frac{1}{6}$ does not.

[5] "Reeling and Writhing, of course, to begin with, "the Mock Turtle replied, "and the different branches of Arithmetic–Ambition, Distraction, Uglification, and Derision."

[6] That this cancellation is meaningful depends on the concept of *convergence*, of which we may talk more later.

[7] The curious reader may find a proof in many a good number theory book, for example [HarWri]

Is every real number a rational number? Enter the Pythagorean Society in the picture, whose founder, Pythagoras lived 582 to 500 BC. This loony sect of Greeks forbade their members to eat beans. But their lunacy went even farther. Rather than studying numbers to solve everyday "real world problems"—as some misguided pedagogues insist—they tried to understand the very essence of numbers, to study numbers in the abstract. At the beginning it seems that they thought that the "only numbers" were rational numbers. But one of them, Hipassos of Metapontum, was able to prove that the length of hypotenuse of a right triangle whose legs[8] had unit length could not be expressed as the ratio of two integers and hence, it was *irrational*.

19 Theorem [Hipassos of Metapontum]$\sqrt{2}$ is irrational.

Proof: *Assume there is* $s \in \mathbb{Q}$ *such that* $s^2 = 2$. *We can find integers* $m, n \neq 0$ *such that* $s = \dfrac{m}{n}$. *The crucial part of the argument is that we can choose* m, n *such that this fraction be in least terms, and hence,* m, n *cannot be both even. Now,* $n^2 s^2 = m^2$, *that is* $2n^2 = m^2$. *This means that* m^2 *is even. But then* m *itself must be even, since the product of two odd numbers is odd. Thus* $m = 2a$ *for some non-zero integer* a *(since* $m \neq 0$*). This means that* $2n^2 = (2a)^2 = 4a^2 \implies n^2 = 2a^2$. *This means once again that* n *is even. But then we have a contradiction, since* m *and* n *were not both even.* ❑

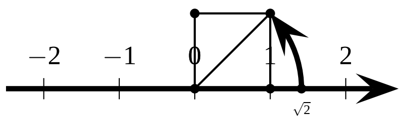

Figure 1.7: Theorem 19.

The above theorem says that the set $\mathbb{R} \setminus \mathbb{Q}$ *of irrational numbers is non-empty. This is one of the very first theorems ever proved. It befits you, dear reader, if you want to be called mathematically literate, to know its proof.*

Suppose that we knew that every strictly positive natural number has a *unique* factorisation into primes. Then if n is not a perfect square we may deduce that, in general, $\sqrt{n}$ is irrational. For, if $\sqrt{n}$ were rational, there would exist two strictly positive natural numbers a, b such that $\sqrt{n} = \dfrac{a}{b}$. This implies that $nb^2 = a^2$. The dextral side of this equality has an even number of prime factors, but the sinistral side does not, since n is not a perfect square. This contradicts unique factorisation, and so $\sqrt{n}$ must be irrational.

From now on we will accept the result that $\sqrt{n}$ *is irrational whenever* n *is a positive non-square integer.*

The shock caused to the other Pythagoreans by Hipassos' result was so great (remember the Pythagoreans were a cult), that they drowned him. Fortunately, mathematicians have matured since then and the task of burning people at the stake or flying planes into skyscrapers has fallen into other hands.

20 Example Give examples, if at all possible, of the following.

1. the sum of two rational numbers giving an irrational number.
2. the sum of two irrationals giving an irrational number.
3. the sum of two irrationals giving a rational number.
4. the product of a rational and an irrational giving an irrational number.
5. the product of a rational and an irrational giving a rational number.
6. the product of two irrationals giving an irrational number.
7. the product of two irrationals giving a rational number.

Solution: ▶

[8] The appropriate word here is "cathetus."

1. *This is impossible. The rational numbers are closed under addition and multiplication.*
2. *Take both numbers to be $\sqrt{2}$. Their sum is $2\sqrt{2}$ which is also irrational.*
3. *Take one number to be $\sqrt{2}$ and the other $-\sqrt{2}$. Their sum is 0, which is rational.*
4. *take the rational number to be 1 and the irrational to be $\sqrt{2}$. Their product is $1\cdot\sqrt{2}=\sqrt{2}$.*
5. *Take the rational number to be 0 and the irrational to be $\sqrt{2}$. Their product is $0\cdot\sqrt{2}=0$.*
6. *Take one irrational number to be $\sqrt{2}$ and the other to be $\sqrt{3}$. Their product is $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$.*
7. *Take one irrational number to be $\sqrt{2}$ and the other to be $\dfrac{1}{\sqrt{2}}$. Their product is $\sqrt{2}\cdot\dfrac{1}{\sqrt{2}}=1$.*

◀

After the discovery that $\sqrt{2}$ was irrational, suspicion arose that there were other irrational numbers. In fact, Archimedes suspected that π was irrational, a fact that wasn't proved till the XIX-th Century by Lambert. The "irrationalities" of $\sqrt{2}$ and π are of two entirely "different flavours," but we will need several more years of mathematical study[9] to even comprehend the meaning of that assertion.

Irrational numbers, that is, the set $\mathbb{R}\setminus\mathbb{Q}$, are those then having infinite non-repeating decimal expansions. Of course, by simply "looking" at the decimal expansion of a number we can't tell whether it is irrational or rational without having more information. Your calculator probably gives about 9 decimal places when you try to compute $\sqrt{2}$, say, it says $\sqrt{2}\approx 1.414213562$. What happens after the final 2 is the interesting question. Do we have a pattern or do we not?

21 Example We expect a number like

$$0.100100001000000001\ldots,$$

where there are $2,4,8,16,\ldots$ zeroes between consecutive ones, to be irrational, since the gaps between successive 1's keep getting longer, and so the decimal does not repeat. For the same reason, the number

$$0.123456789101112\ldots,$$

which consists of enumerating all strictly positive natural numbers after the decimal point, is irrational. This number is known as the *Champernowne-Mahler* number.

22 Example Prove that $\sqrt[4]{2}$ is irrational.

Solution: ► *If $\sqrt[4]{2}$ were rational, then there would be two non-zero natural numbers, a,b such that*

$$\sqrt[4]{2}=\frac{a}{b} \implies \sqrt{2}=\frac{a^2}{b^2}.$$

Since $\dfrac{a}{b}$ is rational, $\dfrac{a^2}{b^2}=\dfrac{a}{b}\cdot\dfrac{a}{b}$ must also be rational. This says that $\sqrt{2}$ is rational, contradicting Theorem 19. ◀

Homework

1.2.1 Problem Write the infinitely repeating decimal $0.\overline{123}=0.123123123\ldots$ as the quotient of two positive integers.

1.2.2 Problem Prove that $\sqrt{8}$ is irrational.

1.2.3 Problem Assuming that $\sqrt{6}$ is irrational, prove that $\sqrt{2}+\sqrt{3}$ must be irrational.

1.2.4 Problem Suppose that you are given a finite string of integers, say, 12345. Can you find an *irrational* number whose first five decimal digits after the decimal point are 12345?

1.2.5 Problem Find a rational number between the irrational numbers $\sqrt{2}$ and $\sqrt{3}$.

1.2.6 Problem Find an irrational number between the irrational numbers $\sqrt{2}$ and $\sqrt{3}$.

[9] Or in the case of people in the English and the Social Sciences Departments, as many lifetimes as a cat.

1.2.7 Problem Find an irrational number between the rational numbers $\frac{1}{10}$ and $\frac{1}{9}$.

1.3 Operations with Real Numbers

The set of real numbers is furnished with two operations $+$ (addition) and $\cdot$ (multiplication) that satisfy the following axioms.

23 Axiom (Closure)

$$x \in \mathbb{R} \quad \text{and} \quad y \in \mathbb{R} \implies x+y \in \mathbb{R} \quad \text{and} \quad xy \in \mathbb{R}.$$

This axiom tells us that if we add or multiply two real numbers, then we stay within the realm of real numbers. Notice that this is not true of division, for, say, $1 \div 0$ is the division of two real numbers, but $1 \div 0$ is not a real number. This is also not true of taking square roots, for, say, -1 is a real number but $\sqrt{-1}$ is not.

24 Axiom (Commutativity)

$$x \in \mathbb{R} \quad \text{and} \quad y \in \mathbb{R} \implies x+y = y+x \quad \text{and} \quad xy = yx.$$

This axiom tells us that order is immaterial when we add or multiply two real numbers. Observe that this axiom does not hold for division, because, for example, $1 \div 2 \neq 2 \div 1$.

25 Axiom (Associativity)

$$x \in \mathbb{R}, y \in \mathbb{R} \quad \text{and} \quad z \in \mathbb{R} \implies x+(y+z) = (x+y)+z \quad \text{and} \quad (xy)z = x(yz).$$

This axiom tells us that in a string of successive additions or multiplications, it is immaterial where we put the parentheses. Observe that subtraction is not associative, since, for example, $(1-1)-1 \neq 1-(1-1)$.

26 Axiom (Additive and Multiplicative Identity) There exist two unique elements, 0 and 1, with $0 \neq 1$, such that $\forall x \in \mathbb{R}$,

$$0+x = x+0 = x, \quad \text{and} \quad 1 \cdot x = x \cdot 1 = x.$$

27 Axiom (Existence of Opposites and Inverses) For all $x \in \mathbb{R}\ \exists -x \in \mathbb{R}$, called the *opposite* of x, such that

$$x+(-x) = (-x)+x = 0.$$

For all $y \in \mathbb{R} \setminus \{0\}\ \exists y^{-1} \in \mathbb{R} \setminus \{0\}$, called the *multiplicative inverse* of y, such that

$$y \cdot y^{-1} = y^{-1} \cdot y = 1.$$

In the axiom above, notice that 0 does not have a multiplicative inverse, that is, division by 0 is not allowed. Why? Let us for a moment suppose that 0 had a multiplicative inverse, say 0^{-1}. We will obtain a contradiction as follows. First, if we multiply any real number by 0 we get 0, so, in particular, $0 \cdot 0^{-1} = 0$. Also, if we multiply a number by its multiplicative inverse we should get 1, and hence, $0 \cdot 0^{-1} = 1$. This gives

$$0 = 0 \cdot 0^{-1} = 1,$$

in contradiction to the assumption that $0 \neq 1$.

28 Axiom (Distributive Law) For all real numbers x, y, z, there holds the equality

$$x \cdot (y+z) = x \cdot y + x \cdot z.$$

STOP *It is customary in Mathematics to express a product like $x \cdot y$ by juxtaposition, that is, by writing together the letters, as in xy, omitting the product symbol $\cdot$. From now on we will follow this custom.*

The above axioms allow us to obtain various algebraic identities, of which we will demonstrate a few.

29 Theorem (Difference of Squares Identity) For all real numbers a, b, there holds the identity

$$a^2 - b^2 = (a-b)(a+b).$$

Proof: *Using the distributive law twice,*

$$(a-b)(a+b) = a(a+b) - b(a+b) = a^2 + ab - ba - b^2 = a^2 + ab - ab - b^2 = a^2 - b^2.$$

❑

Here is an application of the above identity.

30 Example Given that $2^{32} - 1$ has exactly two divisors a and b satisfying the inequalities

$$50 < a < b < 100,$$

find the product ab.

Solution: ► *We have*

$$\begin{aligned} 2^{32} - 1 &= (2^{16}-1)(2^{16}+1) \\ &= (2^8-1)(2^8+1)(2^{16}+1) \\ &= (2^4-1)(2^4+1)(2^8+1)(2^{16}+1) \\ &= (2^2-1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1) \\ &= (2-1)(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1). \end{aligned}$$

Since $2^8 + 1 = 257$, *a and b must be part of the product*

$$(2-1)(2+1)(2^2+1)(2^4+1) = 255 = 3 \cdot 5 \cdot 17.$$

The only divisors of 255 *in the desired range are* $3 \cdot 17 = 51$ *and* $5 \cdot 17 = 85$, *whence the desired product is* $51 \cdot 85 = 4335$. ◄

31 Theorem (Difference and Sum of Cubes) For all real numbers a, b, there holds the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2) \qquad \text{and} \qquad a^3 + b^3 = (a+b)(a^2-ab+b^2).$$

Proof: *Using the distributive law twice,*

$$(a-b)(a^2+ab+b^2) = a(a^2+ab+b^2) - b(a^2+ab+b^2) = a^3 + a^2b + ab^2 - ba^2 - ab^2 - b^3 = a^3 - b^3.$$

Also, replacing b by $-b$ in the difference of cubes identity,

$$a^3 + b^3 = a^3 - (-b)^3 = (a-(-b))(a^2 + a(-b) + (-b)^2) = (a+b)(a^2-ab+b^2).$$

❑

Theorems 29 and 31 can be generalised as follows. Let $n > 0$ be an integer. Then for all real numbers x, y

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1}). \tag{1.1}$$

For example,

$$x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4), \qquad x^5 + y^5 = (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

See problem 1.3.17.

32 Theorem (Perfect Squares Identity) For all real numbers a, b, there hold the identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{and} \qquad (a-b)^2 = a^2 - 2ab + b^2.$$

Proof: *Expanding using the distributive law twice,*

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2.$$

To obtain the second identity, replace b by $-b$ in the just obtained identity:

$$(a-b)^2 = (a+(-b))^2 = a^2 + 2a(-b) + (-b)^2 = a^2 - 2ab + b^2.$$

❑

33 Example The sum of two numbers is 7 and their product is 3. Find the sum of their squares and the sum of their cubes.

Solution: ► *Let the two numbers be a, b. Then $a+b=7$ and $ab=3$. Then*

$$49 = (a+b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 6 \implies a^2 + b^2 = 49 - 6 = 43.$$

Also,

$$a^3 + b^3 = (a+b)(a^2 + b^2 - ab) = (7)(43-3) = 280.$$

Thus the sum of their squares is 43 and the sum of their cubes is 280. ◄

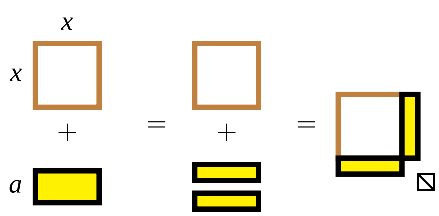

Figure 1.8: Completing the square: $x^2 + ax = \left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$.

The following method, called *Sophie Germain's trick*[10] is useful to convert some expressions into differences of squares.

34 Example We have

$$\begin{aligned} x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 \\ &= (x^2+1)^2 - x^2 \\ &= (x^2 + 1 - x)(x^2 + 1 + x). \end{aligned}$$

35 Example We have

$$\begin{aligned} x^4 + 4 &= x^4 + 4x^2 + 4 - 4x^2 \\ &= (x^2+2)^2 - 4x^2 \\ &= (x^2 + 2 - 2x)(x^2 + 2 + 2x). \end{aligned}$$

[10] Sophie Germain (1776–1831) was an important French mathematician of the French Revolution. She pretended to be a man in order to study Mathematics. At the time, women were not allowed to matriculate at the École Polytechnique, but she posed as a M. Leblanc in order to obtain lessons from Lagrange.

Sophie Germain's trick is often used in factoring quadratic trinomials, where it is often referred to as the technique of *completing the square*, which has the geometric interpretation given in figure 1.8. We will give some examples of factorisations that we may also obtain with the trial an error method commonly taught in elementary algebra.

36 Example We have

$$\begin{aligned} x^2-8x-9 &= x^2-8x+16-9-16 \\ &= (x-4)^2-25 \\ &= (x-4)^2-5^2 \\ &= (x-4-5)(x-4+5) \\ &= (x-9)(x+1). \end{aligned}$$

Here to complete the square, we looked at the coefficient of the linear term, which is -8, we divided by 2, obtaining -4, and then squared, obtaining 16.

37 Example We have

$$\begin{aligned} x^2+4x-117 &= x^2+4x+4-117-4 \\ &= (x+2)^2-11^2 \\ &= (x+2-11)(x+2+11) \\ &= (x-9)(x+13). \end{aligned}$$

Here to complete the square, we looked at the coefficient of the linear term, which is 4, we divided by 2, obtaining 2, and then squared, obtaining 4.

38 Example We have

$$a^2+ab+b^2 = a^2+ab+\frac{b^2}{4}-\frac{b^2}{4}+b^2 = a^2+ab+\frac{b^2}{4}+\frac{3b^2}{4} = \left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}.$$

Here to complete the square, we looked at the coefficient of the linear term (in a), which is b, we divided by 2, obtaining $\frac{b}{2}$, and then squared, obtaining $\frac{b^2}{4}$.

39 Example Factor $2x^2+3x-8$ into linear factors by completing squares.

Solution: ► *First, we force a* 1 *as coefficient of the square term:*

$$2x^2+3x-8 = 2\left(x^2+\frac{3}{2}x-4\right).$$

Then we look at the coefficient of the linear term, which is $\frac{3}{2}$. *We divide it by* 2, *obtaining* $\frac{3}{4}$, *and square it, obtaining* $\frac{9}{16}$. *Hence*

$$\begin{aligned} 2x^2+3x-8 &= 2\left(x^2+\frac{3}{2}x-4\right) \\ &= 2\left(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}-4\right) \\ &= 2\left(\left(x+\frac{3}{2}\right)^2-\frac{9}{16}-4\right) \\ &= 2\left(\left(x+\frac{3}{2}\right)^2-\frac{73}{16}\right) \\ &= 2\left(x+\frac{3}{2}-\frac{\sqrt{73}}{4}\right)\left(x+\frac{3}{2}+\frac{\sqrt{73}}{4}\right). \end{aligned}$$

◀

40 Theorem (Perfect Cubes Identity) For all real numbers a, b, there hold the identities

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3 \qquad \text{and} \qquad (a-b)^3 = a^3-3a^2b+3ab^2-b^3.$$

Proof: *Expanding, using Theorem 32,*

$$\begin{aligned} (a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a(a^2+2ab+b^2)+b(a^2+2ab+b^2) \\ &= a^3+2a^2b+ab^2+ba^2+2ab^2+b^3 \\ &= a^3+3a^2b+3ab^2+b^3. \end{aligned}$$

The second identity is obtained by replacing b with $-b$*:*

$$(a-b)^3 = (a+(-b))^3 = a^2+3a^2(-b)+3a(-b)^2+(-b)^3 = a^3-3a^2b+3ab^2-b^3.$$

❑

It is often convenient to rewrite the above identities as

$$(a+b)^3 = a^3+b^3+3ab(a+b), \qquad (a-b)^3 = a^3-b^3-3ab(a-b).$$

41 Example Redo example 33 using Theorem 40.

Solution: ► *Again, let the two numbers the two numbers* a, b *satisfy* $a+b=7$ *and* $ab=3$. *Then*

$$343 = 7^3 = (a+b)^3 = a^3+b^3+3ab(a+b) = a^3+b^3+3(3)(7) \implies a^3+b^3 = 343-63 = 280,$$

as before. ◀

The results of Theorems 32 and 40 generalise in various ways. In Appendix B we present the binomial theorem, which provides the general expansion of $(a+b)^n$ when n is a positive integer.

Homework

1.3.1 Problem Expand and collect like terms:

$$\left(\frac{2}{x}+\frac{x}{2}\right)^2-\left(\frac{2}{x}-\frac{x}{2}\right)^2.$$

1.3.2 Problem Find all the real solutions to the system of equations

$$x+y=1, \qquad xy=-2.$$

1.3.3 Problem Find all the real solutions to the system of equations

$$x^3+y^3=7, \qquad x+y=1.$$

1.3.4 Problem Compute

$$1^2-2^2+3^2-4^2+\cdots+99^2-100^2.$$

1.3.5 Problem Let $n\in\mathbb{N}$. Find all prime numbers of the form n^3-8.

1.3.6 Problem Compute $1234567890^2-1234567889\cdot 1234567891$ *mentally*.

1.3.7 Problem The sum of two numbers is 3 and their product is 9. What is the sum of their reciprocals?

1.3.8 Problem Given that

$$1,\ 000,\ 002,\ 000,\ 001$$

has a prime factor greater than 9000, find it.

1.3.9 Problem Let a,b,c be arbitrary real numbers. Prove that

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca).$$

1.3.10 Problem Let a,b,c be arbitrary real numbers. Prove that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

1.3.11 Problem The numbers a,b,c satisfy

$$a+b+c=-6, \qquad ab+bc+ca=2, \qquad a^3+b^3+c^3=6.$$

Find abc.

1.3.12 Problem Compute

$$\sqrt{(1000000)(1000001)(1000002)(1000003)+1}$$

without a calculator.

1.3.13 Problem Find two positive integers a,b such that

$$\sqrt{5+2\sqrt{6}}=\sqrt{a}+\sqrt{b}.$$

1.3.14 Problem If a,b,c,d, are real numbers such that

$$a^2+b^2+c^2+d^2=ab+bc+cd+da,$$

prove that $a=b=c=d$.

1.3.15 Problem Find all real solutions to the equation

$$(x+y)^2=(x-1)(y+1).$$

1.3.16 Problem Let a,b,c be real numbers with $a+b+c=0$. Prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}=\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}.$$

1.3.17 Problem Prove that if $a\in\mathbb{R}$, $a\neq 1$ and $n\in\mathbb{N}\setminus\{0\}$, then

$$1+a+a^2+\cdots a^{n-1}=\frac{1-a^n}{1-a}. \tag{1.2}$$

Then deduce that if n is a strictly positive integer, it follows

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}).$$

1.3.18 Problem Prove that the product of two sums of squares is a sum of squares. That is, let a,b,c,d be integers. Prove that you can find integers A,B such that

$$(a^2+b^2)(c^2+d^2)=A^2+B^2.$$

1.3.19 Problem Prove that if a,b,c are real numbers, then

$$(a+b+c)^3-3(a+b)(b+c)(c+a)=a^3+b^3+c^3.$$

1.3.20 Problem If a,b,c are real numbers, prove that $a^5+b^5+c^5$ equals

$$(a+b+c)^5-5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca).$$

1.4 Order on the Line

Vocabulary Alert! We will call a number x positive *if $x\geq 0$ and* strictly positive *if $x>0$. Similarly, we will call a number y* negative *if $y\leq 0$ and* strictly negative *if $y<0$. This usage differs from most Anglo-American books, who prefer such terms as* non-negative *and* non-positive.

The set of real numbers $\mathbb{R}$ is also endowed with a relation $>$ which satisfies the following axioms.

42 Axiom (Trichotomy Law) For all real numbers x, y exactly one of the following holds:

$$x > y, \quad x = y, \quad \text{or} \quad y > x.$$

43 Axiom (Transitivity of Order) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{and} \quad y > z \quad \text{then} \quad x > z.$$

44 Axiom (Preservation of Inequalities by Addition) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{then} \quad x + z > y + z.$$

45 Axiom (Preservation of Inequalities by Positive Factors) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{and} \quad z > 0 \quad \text{then} \quad xz > yz.$$

46 Axiom (Inversion of Inequalities by Negative Factors) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{and} \quad z < 0 \quad \text{then} \quad xz < yz.$$

STOP *$x < y$ means that $y > x$. $x \leq y$ means that either $y > x$ or $y = x$, etc.*

The above axioms allow us to solve several inequality problems.

47 Example Solve the inequality

$$2x - 3 < -13.$$

Solution: ► *We have*

$$2x - 3 < -13 \implies 2x < -13 + 3 \implies 2x < -10.$$

The next step would be to divide both sides by 2. *Since* $2 > 0$, *the sense of the inequality is preserved, whence*

$$2x < -10 \implies x < \frac{-10}{2} \implies x < -5.$$

◄

48 Example Solve the inequality

$$-2x - 3 \leq -13.$$

Solution: ► *We have*

$$-2x - 3 \leq -13 \implies -2x \leq -13 + 3 \implies -2x \leq -10.$$

The next step would be to divide both sides by -2. *Since* $-2 < 0$, *the sense of the inequality is inverted, and so*

$$-2x \leq -10 \implies x \geq \frac{-10}{-2} \implies x \geq -5.$$

◄

The method above can be generalised for the case of a product of linear factors. To investigate the set on the line where the inequality

$$(a_1x + b_1)\cdots(a_nx + b_n) > 0, \tag{1.3}$$

holds, we examine each individual factor. By trichotomy, for every k, the real line will be split into the three distinct zones

$$\{x \in \mathbb{R} : a_kx + b_k > 0\} \cup \{x \in \mathbb{R} : a_kx + b_k = 0\} \cup \{x \in \mathbb{R} : a_kx + b_k < 0\}.$$

We will call the real line with punctures at $x = -\dfrac{a_k}{b_k}$ and indicating where each factor changes sign the *sign diagram* corresponding to the inequality (1.3).

49 Example Consider the inequality
$$x^2+2x-35<0.$$

1. Form a sign diagram for this inequality.
2. Write the set $\{x\in\mathbb{R} : x^2+2x-35<0\}$ as an interval or as a union of intervals.
3. Write the set $\left\{x\in\mathbb{R} : x^2+2x-35\geq 0\right\}$ as an interval or as a union of intervals.
4. Write the set $\left\{x\in\mathbb{R} : \dfrac{x+7}{x-5}\geq 0\right\}$ as an interval or as a union of intervals.
5. Write the set $\left\{x\in\mathbb{R} : \dfrac{x+7}{x-5}\leq -2\right\}$ as an interval or as a union of intervals.

Solution: ►

1. *Observe that* $x^2+2x-35=(x-5)(x+7)$, *which vanishes when* $x=-7$ *or when* $x=5$. *In neighbourhoods of* $x=-7$ *and of* $x=5$, *we find:*

$x\in$	$]-\infty;-7[$	$]-7;5[$	$]5;+\infty[$
$x+7$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
$(x+7)(x-5)$	$+$	$-$	$+$

On the last row, the sign of the product $(x+7)(x-5)$ *is determined by the sign of each of the factors* $x+7$ *and* $x-5$.

2. *From the sign diagram above we see that*
$$\{x\in\mathbb{R} : x^2+2x-35<0\}=\,]-7;5[\,.$$

3. *From the sign diagram above we see that*
$$\left\{x\in\mathbb{R} : x^2+2x-35\geq 0\right\}=\,]-\infty;-7]\cup[5;+\infty[\,.$$
Notice that we include both $x=-7$ *and* $x=5$ *in the set, as* $(x+7)(x-5)$ *vanishes there.*

4. *From the sign diagram above we see that*
$$\left\{x\in\mathbb{R} : \frac{x+7}{x-5}\geq 0\right\}=\,]-\infty;-7]\cup]5;+\infty[\,.$$
Notice that we include $x=-7$ *since* $\dfrac{x+7}{x-5}$ *vanishes there, but we do not include* $x=5$ *since there the fraction* $\dfrac{x+7}{x-5}$ *would be undefined.*

5. *We must add fractions:*
$$\frac{x+7}{x-5}\leq -2 \iff \frac{x+7}{x-5}+2\leq 0 \iff \frac{x+7}{x-5}+\frac{2x-10}{x-5}\leq 0 \iff \frac{3x-3}{x-5}\leq 0.$$
We must now construct a sign diagram puncturing the line at $x=1$ *and* $x=5$:

$x\in$	$]-\infty;1[$	$]1;5[$	$]5;+\infty[$
$3x-3$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
$\dfrac{3x-3}{x-5}$	$+$	$-$	$+$

We deduce that

$$\left\{x \in \mathbb{R} : \frac{x+7}{x-5} \leq -2\right\} = [1;5[.$$

Notice that we include $x = 1$ *since* $\dfrac{3x-3}{x-5}$ *vanishes there, but we exclude* $x = 5$ *since there the fraction* $\dfrac{3x-3}{x-5}$ *is undefined.*

◀

50 Example Determine the following set explicitly: $\{x \in \mathbb{R} : -x^2 + 2x - 2 \geq 0\}$.

Solution: ▶ *The equation* $-x^2 + 2x - 2 = 0$ *does not have rational roots. To find its roots we either use the quadratic formula, or we may complete squares. We will use the latter method:*

$$-x^2 + 2x - 2 = -(x^2 - 2x) - 2 = -(x^2 - 2x + 1) - 2 + 1 = -(x-1)^2 - 1.$$

Therefore,

$$-x^2 + 2x - 2 \geq 0 \iff -(x-1)^2 - 1 \geq 0 \iff -((x-1)^2 + 1) \geq 0.$$

This last inequality is impossible for real numbers, as the expression $-((x-1)^2 + 1)$ *is strictly negative. Hence,*

$$\{x \in \mathbb{R} : -x^2 + 2x - 2 \geq 0\} = \varnothing.$$

Aliter: *The discriminant of* $-x^2 + 2x - 2$ *is* $2^2 - 4(-1)(-2) = -4 < 0$, *which means that the equation has complex roots. Hence the quadratic polynomial keeps the sign of its leading coefficient,* -1, *and so it is always negative.*
◀

51 Example Determine explicitly the set $\{x \in \mathbb{R} : 32x^2 - 40x + 9 > 0\}$.

Solution: ▶ *The equation* $32x^2 - 40x + 9 = 0$ *does not have rational roots. To find its roots we will complete squares:*

$$\begin{aligned} 32x^2 - 40x + 9 &= 32\left(x^2 - \frac{5}{4}x + \frac{9}{32}\right) \\ &= 32\left(x^2 - \frac{5}{4}x + \frac{5^2}{8^2} + \frac{9}{32} - \frac{5^2}{8^2}\right) \\ &= 32\left(\left(x - \frac{5}{8}\right)^2 - \frac{7}{64}\right) \\ &= 32\left(x - \frac{5}{8} - \frac{\sqrt{7}}{8}\right)\left(x - \frac{5}{8} + \frac{\sqrt{7}}{8}\right). \end{aligned}$$

We may now form a sign diagram, puncturing the line at $x = \dfrac{5}{8} - \dfrac{\sqrt{7}}{8}$ *and at* $x = \dfrac{5}{8} + \dfrac{\sqrt{7}}{8}$*:*

$x \in$	$\left]-\infty; \frac{5}{8} - \frac{\sqrt{7}}{8}\right[$	$\left]\frac{5}{8} - \frac{\sqrt{7}}{8}; \frac{5}{8} + \frac{\sqrt{7}}{8}\right[$	$\left]\frac{5}{8} + \frac{\sqrt{7}}{8}; +\infty\right[$
$\left(x - \frac{5}{8} + \frac{\sqrt{7}}{8}\right)$	$-$	$+$	$+$
$\left(x - \frac{5}{8} - \frac{\sqrt{7}}{8}\right)$	$-$	$-$	$+$
$\left(x - \frac{5}{8} + \frac{\sqrt{7}}{8}\right)\left(x - \frac{5}{8} - \frac{\sqrt{7}}{8}\right)$	$+$	$-$	$+$

We deduce that

$$\left\{x \in \mathbb{R} : 32x^2 - 40x + 9 > 0\right\} = \left]-\infty; \frac{5}{8} - \frac{\sqrt{7}}{8}\right[\cup \left]\frac{5}{8} + \frac{\sqrt{7}}{8}; +\infty\right[.$$

◀

Care must be taken when transforming an inequality, as a given transformation may introduce spurious solutions.

52 Example Solve the inequality
$$2\sqrt{1-x}-\sqrt{x+1}\geq\sqrt{x}.$$

Solution: ► *For the square roots to make sense, we must have*
$$x\in\,]-\infty;1]\cap[-1;+\infty[\cap[0;+\infty[\implies x\in[0;1].$$
Squaring both sides of the inequality, transposing, and then squaring again,
$$4(1-x)-4\sqrt{1-x^2}+x+1>x\implies 5-4x>4\sqrt{1-x^2}\implies 25-40x+16x^2>16-16x^2\implies 32x^2-40x+9>0.$$
This last inequality has already been solved in example 51. Thus we want the intersection
$$\left(\left]-\infty;\frac{5}{8}-\frac{\sqrt{7}}{8}\right[\cup\left]\frac{5}{8}+\frac{\sqrt{7}}{8};+\infty\right[\right)\cap[0;1]=\left[0;\frac{5}{8}-\frac{\sqrt{7}}{8}\right[.$$
◄

Homework

1.4.1 Problem Consider the set
$$\{x\in\mathbb{R}:x^2-x-6\leq 0\}.$$

1. Draw a sign diagram for this set.
2. Using the obtained sign diagram, write the set
$$\{x\in\mathbb{R}:x^2-x-6\leq 0\}$$
as an interval or as a union of intervals.
3. Using the obtained sign diagram, write the set
$$\left\{x\in\mathbb{R}:\frac{x-3}{x+2}\geq 0\right\}$$
as an interval or as a union of intervals.

1.4.2 Problem Write the set
$$\left\{x\in\mathbb{R}:\frac{x^2+x-6}{x^2-x-6}\geq 0\right\}$$
as an interval or as a union of intervals.

1.4.3 Problem Give an explicit description of the set
$$\{x\in\mathbb{R}:x^2-x-4\geq 0\}.$$

1.4.4 Problem Write the set
$$\left\{x\in\mathbb{R}:x^2-x-6\leq 0\right\}\cap\left\{x\in\mathbb{R}:\frac{1-x}{x+3}\geq 1\right\}$$
in interval notation.

1.4.5 Problem Solve the inequality $\sqrt{x^2-4x+3}\geq -x+2$.

1.4.6 Problem Solve the inequality $\dfrac{1-\sqrt{1-4x^2}}{x}>\dfrac{1}{2}$.

1.4.7 Problem Solve the inequality
$$\sqrt{2x+1}+\sqrt{2x-5}\geq\sqrt{5-2x}.$$

1.4.8 Problem Find the least positive integer n satisfying the inequality
$$\sqrt{n+1}-\sqrt{n}<\frac{1}{10}.$$

1.4.9 Problem Determine the values of the real parameter t such that the set
$$A_t=\left\{x\in\mathbb{R}:(t-1)x^2+tx+\frac{t}{4}=0\right\}$$

1. be empty;
2. have exactly one element;
3. have exactly two elements.

1.4.10 Problem List the elements of the set
$$\left\{x\in\mathbb{Z}:\min\left(x+2,4-\frac{x}{3}\right)\geq 1\right\}.$$

1.4.11 Problem Demonstrate that for all real numbers $x>0$ it is verified that
$$2x^3-6x^2+\frac{11}{2}x+1>0.$$

1.4.12 Problem Demonstrate that for all real numbers x it is verified that
$$x^8-x^5+x^2-x+1>0.$$

1.4.13 Problem The values of a,b,c, and d are $1,2,3$ and 4 but not necessarily in that order. What is the largest possible value of $ab+bc+cd+da$?

1.4.14 Problem Prove that if $r\geq s\geq t$ then
$$r^2-s^2+t^2\geq(r-s+t)^2.$$

1.5 Absolute Value

We start with a definition.

53 Definition Let $x \in \mathbb{R}$. The *absolute value of x*—denoted by $|x|$—is defined by

$$\boxed{|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}}$$

The absolute value of a real number is thus the distance of that real number to 0, and hence $|x-y|$ is the distance between x and y on the real line. The absolute value of a quantity is either the quantity itself or its opposite.

54 Example Write without absolute value signs:

1. $|\sqrt{3}-2|$,
2. $|\sqrt{7}-\sqrt{5}|$,
3. $||\sqrt{7}-\sqrt{5}|-|\sqrt{3}-2||$

Solution: ► *We have*

1. *since* $2 > 1.74 > \sqrt{3}$, *we have* $|\sqrt{3}-2| = 2-\sqrt{3}$.
2. *since* $\sqrt{7} > \sqrt{5}$, *we have* $|\sqrt{7}-\sqrt{5}| = \sqrt{7}-\sqrt{5}$.
3. *by virtue of the above calculations,*

$$||\sqrt{7}-\sqrt{5}|-|\sqrt{3}-2|| = |\sqrt{7}-\sqrt{5}-(2-\sqrt{3})| = |\sqrt{7}+\sqrt{3}-\sqrt{5}-2|.$$

The question we must now answer is whether $\sqrt{7}+\sqrt{3} > \sqrt{5}+2$. *But* $\sqrt{7}+\sqrt{3} > 4.38 > \sqrt{5}+2$ *and hence*

$$|\sqrt{7}+\sqrt{3}-\sqrt{5}-2| = \sqrt{7}+\sqrt{3}-\sqrt{5}-2.$$

◄

55 Example Let $x > 10$. Write $|3-|5-x||$ without absolute values.

Solution: ► *We know that* $|5-x| = 5-x$ *if* $5-x \geq 0$ *or that* $|5-x| = -(5-x)$ *if* $5-x < 0$. *As* $x > 10$, *we have then* $|5-x| = x-5$. *Therefore*

$$|3-|5-x|| = |3-(x-5)| = |8-x|.$$

In the same manner , either $|8-x| = 8-x$ *if* $8-x \geq 0$ *or* $|8-x| = -(8-x)$ *if* $8-x < 0$. *As* $x > 10$, *we have then* $|8-x| = x-8$. *We conclude that* $x > 10$,

$$|3-|5-x|| = x-8.$$

◄

The method of sign diagrams from the preceding section is also useful when considering expressions involving absolute values.

56 Example Find all real solutions to $|x+1|+|x+2|-|x-3| = 5$.

Solution: ► *The vanishing points for the absolute value terms are* $x = -1$, $x = -2$ *and* $x = 3$. *Notice that these are the points where the absolute value terms change sign. We decompose* $\mathbb{R}$ *into (overlapping) intervals with endpoints at the places where each of the expressions in absolute values vanish. Thus we have*

$$\mathbb{R} =]-\infty; -2] \cup [-2; -1] \cup [-1; 3] \cup [3; +\infty[.$$

We examine the sign diagram

$x \in$	$]-\infty;-2]$	$[-2;-1]$	$[-1;3]$	$[3;+\infty[$
$\lvert x+2\rvert =$	$-x-2$	$x+2$	$x+2$	$x+2$
$\lvert x+1\rvert =$	$-x-1$	$-x-1$	$x+1$	$x+1$
$\lvert x-3\rvert =$	$-x+3$	$-x+3$	$-x+3$	$x-3$
$\lvert x+2\rvert+\lvert x+1\rvert-\lvert x-3\rvert =$	$-x-6$	$x-2$	$3x$	$x+6$

Thus on $]-\infty;-2]$ *we need* $-x-6=5$ *from where* $x=-11$. *On* $[-2;-1]$ *we need* $x-2=5$ *meaning that* $x=7$. *Since* $7 \not\in [-2;-1]$, *this solution is spurious. On* $[-1;3]$ *we need* $3x=5$, *and so* $x=\frac{5}{3}$. *On* $[3;+\infty[$ *we need* $x+6=5$, *giving the spurious solution* $x=-1$. *Upon assembling all this, the solution set is*

$$\left\{-11, \frac{5}{3}\right\}.$$

◀

We will now demonstrate two useful theorems for dealing with inequalities involving absolute values.

57 Theorem Let $t \geq 0$. Then

$$|x| \leq t \iff -t \leq x \leq t.$$

Proof: *Either* $|x|=x$, *or* $|x|=-x$.
If $|x|=x$,

$$|x| \leq t \iff x \leq t \iff -t \leq 0 \leq x \leq t.$$

If $|x|=-x$,

$$|x| \leq t \iff -x \leq t \iff -t \leq x \leq 0 \leq t.$$

❑

58 Example Solve the inequality $|2x-1| \leq 1$.

Solution: ► *From theorem 57*,

$$|2x-1| \leq 1 \iff -1 \leq 2x-1 \leq 1 \iff 0 \leq 2x \leq 2 \iff 0 \leq x \leq 1 \iff x \in [0;1].$$

The solution set is the interval $[0;1]$. ◀

59 Theorem Let $t \geq 0$. Then

$$|x| \geq t \iff x \geq t \quad \text{or} \quad x \leq -t.$$

Proof: *Either* $|x|=x$, *or* $|x|=-x$.
If $|x|=x$,

$$|x| \geq t \iff x \geq t.$$

If $|x|=-x$,

$$|x| \geq t \iff -x \geq t \iff x \leq -t.$$

❑

60 Example Solve the inequality $|3+2x| \geq 1$.

Solution: ► *From theorem 59 ,*

$$|3+2x| \geq 1 \implies 3+2x \geq 1 \quad \text{or} \quad 3+2x \leq -1 \implies x \geq -1 \quad \text{or} \quad x \leq -2.$$

The solution set is the union of intervals $]-\infty;-2]\cup[-1;+\infty[$. ◄

61 Example Solve the inequality $|1-|1-x|| \geq 1$.

Solution: ► *We have*

$$|1-|1-x|| \geq 1 \iff 1-|1-x| \geq 1 \quad \text{or} \quad 1-|1-x| \leq -1.$$

Solving the first inequality,

$$1-|1-x| \geq 1 \iff -|1-x| \geq 0 \implies x = 1,$$

since the quantity $-|1-x|$ *is always negative.*
Solving the second inequality,

$$1-|1-x| \leq -1 \iff -|1-x| \leq -2 \iff |1-x| \geq 2 \iff 1-x \geq 2 \quad \text{or} \quad 1-x \leq -2 \implies x \in [3;+\infty[\cup]-\infty;-1]$$

and thus

$$\{x \in \mathbb{R} : |1-|1-x|| \geq 1\} =]-\infty;-1]\cup\{1\}\cup[3;+\infty[.$$

◄

We conclude this section with a classical inequality involving absolute values.

62 Theorem (Triangle Inequality) Let a, b be real numbers. Then

$$\boxed{|a+b| \leq |a|+|b|.} \tag{1.4}$$

Proof: *Since clearly* $-|a| \leq a \leq |a|$ *and* $-|b| \leq b \leq |b|$, *from Theorem 57, by addition,*

$$-|a| \leq a \leq |a|$$

to

$$-|b| \leq b \leq |b|$$

we obtain

$$-(|a|+|b|) \leq a+b \leq (|a|+|b|),$$

whence the theorem follows. ❑

63 Corollary Let a, b be real numbers. Then

$$\boxed{||a|-|b|| \leq |a-b|}. \tag{1.5}$$

Proof: *We have*

$$|a| = |a-b+b| \leq |a-b|+|b|,$$

giving

$$|a|-|b| \leq |a-b|.$$

Similarly,

$$|b| = |b-a+a| \leq |b-a|+|a| = |a-b|+|a|,$$

gives

$$|b|-|a| \leq |a-b|.$$

The stated inequality follows from this. ❑

Homework

1.5.1 Problem Write without absolute values: $|\sqrt{3}-\sqrt{|2-\sqrt{15}|}\,|$

1.5.2 Problem Write without absolute values if $x>2$: $|x-|1-2x||$.

1.5.3 Problem If $x<-2$ prove that $|1-|1+x||=-2-x$.

1.5.4 Problem Let a,b be real numbers. Prove that $|ab|=|a||b|$.

1.5.5 Problem Let $a\in\mathbb{R}$. Prove that $\sqrt{a^2}=|a|$.

1.5.6 Problem Let $a\in\mathbb{R}$. Prove that $a^2=|a|^2=|a^2|$.

1.5.7 Problem Solve the inequality $|1-2x|<3$.

1.5.8 Problem How many real solutions are there to the equation

$$|x^2-4x|=3\,?$$

1.5.9 Problem Solve the following absolute value equations:

1. $|x-3|+|x+2|=3$,
2. $|x-3|+|x+2|=5$,
3. $|x-3|+|x+2|=7$.

1.5.10 Problem Find all the real solutions of the equation

$$x^2-2|x+1|-2=0.$$

1.5.11 Problem Find all the real solutions to $|5x-2|=|2x+1|$.

1.5.12 Problem Find all real solutions to $|x-2|+|x-3|=1$.

1.5.13 Problem Find the set of solutions to the equation

$$|x|+|x-1|=2.$$

1.5.14 Problem Find the solution set to the equation

$$|x|+|x-1|=1.$$

1.5.15 Problem Find the solution set to the equation

$$|2x|+|x-1|-3|x+2|=1.$$

1.5.16 Problem Find the solution set to the equation

$$|2x|+|x-1|-3|x+2|=-7.$$

1.5.17 Problem Find the solution set to the equation

$$|2x|+|x-1|-3|x+2|=7.$$

1.5.18 Problem If $x<0$ prove that $\left|x-\sqrt{(x-1)^2}\right|=1-2x$.

1.5.19 Problem Find the real solutions, if any, to $|x^2-3x|=2$.

1.5.20 Problem Find the real solutions, if any, to $x^2-2|x|+1=0$.

1.5.21 Problem Find the real solutions, if any, to $x^2-|x|-6=0$.

1.5.22 Problem Find the real solutions, if any, to $x^2=|5x-6|$.

1.5.23 Problem Prove that if $x\leq-3$, then $|x+3|-|x-4|$ is constant.

1.5.24 Problem Solve the equation

$$\left|\frac{2x}{x-1}\right|=|x+1|.$$

1.5.25 Problem Write the set

$$\{x\in\mathbb{R}:|x+1|-|x-2|=-3\}$$

in interval notation.

1.5.26 Problem Let x,y real numbers. Demonstrate that the maximum and the minimum of x and y are given by

$$\max(x,y)=\frac{x+y+|x-y|}{2}$$

and

$$\min(x,y)=\frac{x+y-|x-y|}{2}.$$

1.5.27 Problem Solve the inequality $|x-1||x+2|>4$.

1.5.28 Problem Solve the inequality $\dfrac{|2x^2-1|}{x^2-x-2}>\dfrac{1}{2}$.

1.6 Completeness Axiom

The alert reader may have noticed that the smaller set of rational numbers satisfies all the arithmetic axioms and order axioms of the real numbers given in the preceding sections. Why then, do we need the larger set $\mathbb{R}$? In this section we will present an axiom that characterises the real numbers.

64 Definition A number u is an *upper bound* for a set of numbers A if for all $a\in A$ we have $a\leq u$. The smallest such upper bound is called the *supremum* of the set A. Similarly, a number l is a *lower bound* for a set of numbers B if for all $b\in B$ we have $l\leq b$. The largest such lower bound is called the *infimum* of the set B.

The real numbers have the following property, which further distinguishes them from the rational numbers.

65 Axiom (Completeness of $\mathbb{R}$) Any set of real numbers which is bounded above has a supremum. Any set of real numbers which is bounded below has a infimum.

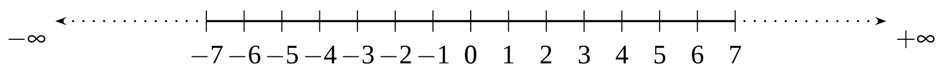

Figure 1.9: The Real Line.

Observe that the rational numbers are not complete. For example, there is no largest rational number in the set

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

since $\sqrt{2}$ is irrational and for any good rational approximation to $\sqrt{2}$ we can always find a better one.

Geometrically, each real number can be viewed as a point on a straight line. We make the convention that we orient the real line with 0 as the origin, the positive numbers increasing towards the right from 0 and the negative numbers decreasing towards the left of 0, as in figure 1.9. The Completeness Axiom says, essentially, that this line has no "holes."

We append the object $+\infty$, which is larger than any real number, and the object $-\infty$, which is smaller than any real number. Letting $x \in \mathbb{R}$, we make the following conventions.

$$(+\infty) + (+\infty) = +\infty \tag{1.6}$$

$$(-\infty) + (-\infty) = -\infty \tag{1.7}$$

$$x + (+\infty) = +\infty \tag{1.8}$$

$$x + (-\infty) = -\infty \tag{1.9}$$

$$x(+\infty) = +\infty \quad \text{if } x > 0 \tag{1.10}$$

$$x(+\infty) = -\infty \quad \text{if } x < 0 \tag{1.11}$$

$$x(-\infty) = -\infty \quad \text{if } x > 0 \tag{1.12}$$

$$x(-\infty) = +\infty \quad \text{if } x < 0 \tag{1.13}$$

$$\frac{x}{\pm\infty} = 0 \tag{1.14}$$

Observe that we leave the following undefined:

$$\frac{\pm\infty}{\pm\infty}, \quad (+\infty) + (-\infty), \quad 0(\pm\infty).$$

2 The Plane

2.1 Sets on the Plane

66 Definition Let $A, B,$ be subsets of real numbers. Their *Cartesian Product* $A \times B$ is defined and denoted by

$$A \times B = \{(a,b) : a \in A, b \in B\},$$

that is, the set of all ordered pairs whose elements belong to the given sets.

In the particular case when $A = B$ we write

$$A \times A = A^2.$$

67 Example If $A = \{-1,-2\}$ and $B = \{-1,2\}$ then

$$A \times B = \{(-1,-1),(-1,2),(-2,-1),(-2,2))\},$$

$$B \times A = \{(-1,-1),(-1,-2),(2,-1),(2,-2)\},$$
$$A^2 = \{(-1,-1),(-1,-2),(-2,-1),(-2,-2)\},$$
$$B^2 = \{(-1,-1),(-1,2),(2,-1),(2,2)\}.$$

Notice that these sets are all different, even though some elements are shared.

68 Example $(-1,2) \in \mathbb{Z}^2$ but $(-1,\sqrt{2}) \notin \mathbb{Z}^2$.

69 Example $(-1,\sqrt{2}) \in \mathbb{Z} \times \mathbb{R}$ but $(-1,\sqrt{2}) \notin \mathbb{R} \times \mathbb{Z}$.

70 Definition $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$—the *real Cartesian Plane*—- is the set of all ordered pairs (x,y) of real numbers.

We represent the elements of $\mathbb{R}^2$ graphically as follows. Intersect perpendicularly two copies of the real number line. These two lines are the *axes*. Their point of intersection—which we label $O = (0,0)$— is the *origin*. To each point P on the plane we associate an ordered pair $P = (x,y)$ of real numbers. Here x is the *abscissa*[1], which measures the horizontal distance of our point to the origin, and y is the *ordinate*, which measures the vertical distance of our point to the origin. The points x and y are the *coordinates* of P. This manner of dividing the plane and labelling its points is called the *Cartesian coordinate system.* The horizontal axis is called the *x-axis* and the vertical axis is called the *y-axis*. It is therefore sufficient to have two numbers x and y to completely characterise the position of a point $P = (x,y)$ on the plane $\mathbb{R}^2$.

71 Definition Let $a \in \mathbb{R}$ be a constant. The set

$$\{(x,y) \in \mathbb{R}^2 : x = a\}$$

is a vertical line.

72 Definition Let $b \in \mathbb{R}$ be a constant. The set

$$\{(x,y) \in \mathbb{R}^2 : y = b\}$$

is a horizontal line.

[1] From the Latin *linea abscissa* or line cut-off.

Figures 2.1 and 2.2 give examples of vertical and horizontal lines.

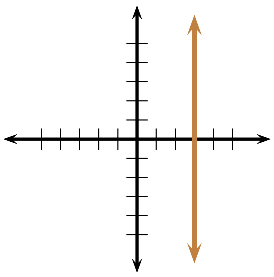

Figure 2.1: Line $x = 3$.

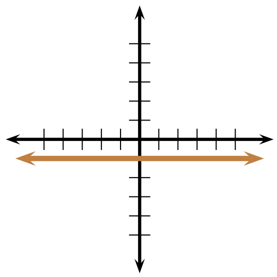

Figure 2.2: Line $y = -1$.

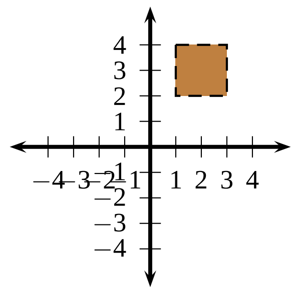

Figure 2.3: Example 74.

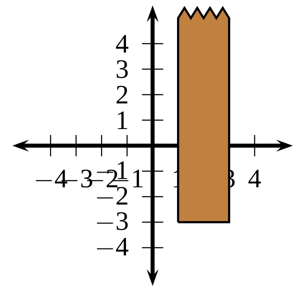

Figure 2.4: Example 75

73 Example Draw the Cartesian product of intervals

$$\mathscr{R} =]1;3[\times]2;4[= \{(x,y) \in \mathbb{R}^2 : 1 < x < 3, \quad 2 < y < 4\}.$$

Solution: ▶ *The set is bounded on the left by the vertical line $x = 1$ and bounded on the right by the vertical line $x = 3$, excluding the lines themselves. The set is bounded above by the horizontal line $y = 4$ and below by the horizontal line $y = 2$, excluding the lines themselves. The set is thus a square minus its boundary, as in figure 2.3.* ◀

74 Example Sketch the region

$$\mathscr{R} = \{(x,y) \in \mathbb{R}^2 : 1 < x < 3, \quad 2 < y < 4\}.$$

Solution: ▶ *The region is a square, excluding its boundary. The graph is shewn in figure 2.3, where we have dashed the boundary lines in order to represent their exclusion.* ◀

75 Example The region

$$\mathscr{R} = [1;3] \times [-3;+\infty[$$

is the infinite half strip on the plane sketched in figure 2.4. The boundary lines are solid, to indicate their inclusion. The upper boundary line is toothed, to indicate that it continues to infinity.

76 Example A quadrilateral has vertices at $A = (5,-9), B = (2,3)$, $C = (0,2)$, and $D = (-8,4)$. Find the area, in square units, of quadrilateral $ABCD$.

Solution: ▶ *Enclose quadrilateral ABCD in right $\triangle AED$, and draw lines parallel to the y-axis in order to form trapezoids $AEFB$, $FBCG$, and right $\triangle GCD$, as in figure 2.5. The area $[ABCD]$ of quadrilateral ABCD is thus given by*

$$\begin{aligned}
[ABCD] &= [AED] - [AEFB] - [FBCG] - [GCD] \\
&= \tfrac{1}{2}(AE)(DE) - \tfrac{1}{2}(FE)(FB + AE) - \\
&\qquad -\tfrac{1}{2}(GF)(GC + FB) - \tfrac{1}{2}(DG)(GC) \\
&= \tfrac{1}{2}(13)(13) - \tfrac{1}{2}(3)(13+1) - \tfrac{1}{2}(2)(2+1) - \tfrac{1}{2}(8)(2) \\
&= 84.5 - 21 - 3 - 8 \\
&= 52.5.
\end{aligned}$$

◀

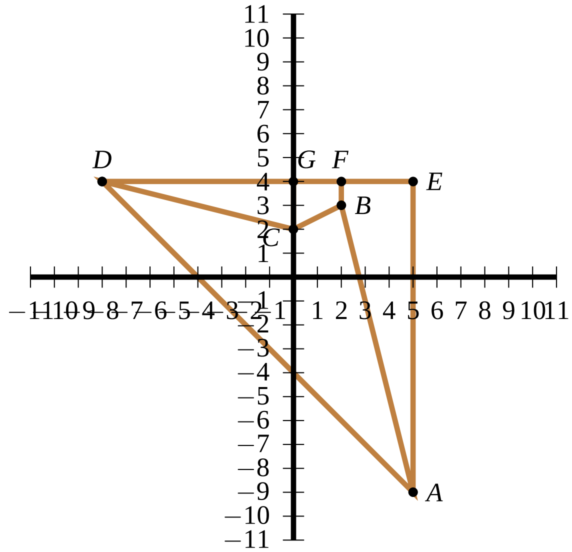

Figure 2.5: Example 76.

Homework

2.1.1 Problem Sketch the following regions on the plane.

1. $R_1 = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$
2. $R_2 = \{(x,y) \in \mathbb{R}^2 : y \geq -3\}$
3. $R_3 = \{(x,y) \in \mathbb{R}^2 : |x| \leq 3, |y| \leq 4\}$
4. $R_4 = \{(x,y) \in \mathbb{R}^2 : |x| \leq 3, |y| \geq 4\}$
5. $R_5 = \{(x,y) \in \mathbb{R}^2 : x \leq 3, y \geq 4\}$
6. $R_6 = \{(x,y) \in \mathbb{R}^2 : x \leq 3, y \leq 4\}$

2.1.2 Problem Find the area of $\triangle ABC$ where $A = (-1,0)$, $B = (0,4)$ and $C = (1,-1)$.

2.1.3 Problem Let $A = [-10;5], B = \{5,6,11\}$ and $C =]-\infty;6[$. Answer the following true or false.

1. $5 \in A$.
2. $6 \in C$.
3. $(0,5,3) \in A \times B \times C$.
4. $(0,-5,3) \in A \times B \times C$.
5. $(0,5,3) \in C \times B \times C$.
6. $A \times B \times C \subseteq C \times B \times C$.
7. $A \times B \times C \subseteq C^3$.

2.1.4 Problem True or false: $(\mathbb{R} \setminus \{0\})^2 = \mathbb{R}^2 \setminus \{(0,0)\}$.

2.2 Distance on the Real Plane

In this section we will deduce some important formulæ from analytic geometry. Our main tool will be the Pythagorean Theorem from elementary geometry.

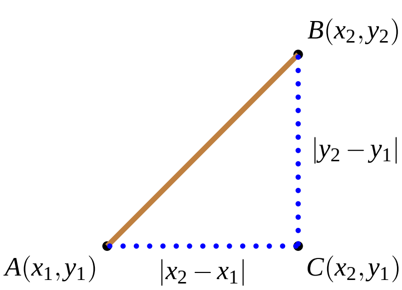

Figure 2.6: Distance between two points.

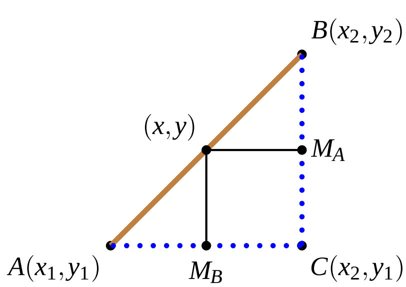

Figure 2.7: Midpoint of a line segment.

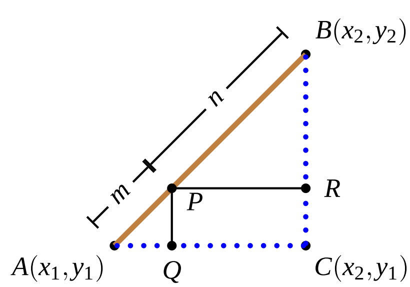

Figure 2.8: Division of a segment.

77 Theorem (Distance Between Two Points on the Plane) The distance between the points $A = (x_1, y_1), B = (x_2, y_2)$ in $\mathbb{R}^2$ is given by

$$\boxed{AB = \mathbf{d}\langle (x_1, y_1), (x_2, y_2)\rangle := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.}$$

Proof: *Consider two points on the plane, as in figure 2.6. Constructing the segments CA and BC with $C = (x_2, y_1)$, we may find the length of the segment AB, that is, the distance from A to B, by utilising the Pythagorean Theorem:*

$$AB^2 = AC^2 + BC^2 \implies AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

❑

78 Example The point $(x, 1)$ is at distance $\sqrt{11}$ from the point $(1, -x)$. Find all the possible values of x.

Solution: ► *We have,*

$$\begin{array}{rrcl} & \mathbf{d}\langle (x,1),(1,-x)\rangle & = & \sqrt{11} \\ \iff & \sqrt{(x-1)^2 + (1+x)^2} & = & \sqrt{11} \\ \iff & (x-1)^2 + (1+x)^2 & = & 11 \\ \iff & 2x^2 + 2 & = & 11. \end{array}$$

Hence, $x = -\frac{3\sqrt{2}}{2}$ *or* $x = \frac{3\sqrt{2}}{2}$. ◄

79 Example Find the point equidistant from $A = (-1, 3), B = (2, 4)$ and $C = (1, 1)$.

Solution: ► *Let (x, y) be the point sought. Then*

$$\mathbf{d}\langle (x,y),(-1,3)\rangle = \mathbf{d}\langle (x,y),(2,4)\rangle \implies (x+1)^2 + (y-3)^2 = (x-2)^2 + (y-4)^2,$$

$$\mathbf{d}\langle (x,y),(-1,3)\rangle = \mathbf{d}\langle (x,y),(1,1)\rangle \implies (x+1)^2 + (y-3)^2 = (x-1)^2 + (y-1)^2.$$

Expanding, we obtain the following linear equations:

$$2x + 1 - 6y + 9 = -4x + 4 - 8y + 16,$$

$$2x + 1 - 6y + 9 = -2x + 1 - 2y + 1,$$

or

$$6x + 2y = 10,$$

$$4x - 4y = -8.$$

We easily find that $(x, y) = \left(\frac{3}{4}, \frac{11}{4}\right)$. ◄

80 Example We say that a point $(x, y) \in \mathbb{R}^2$ is a *lattice point* if $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Demonstrate that no equilateral triangle on the plane may have its three vertices as lattice points.

Solution: ► *Since a triangle may be translated with altering its angles, we may assume, without loss of generality, that we are considering $\triangle ABC$ with $A(0,0)$, $B(b,0)$, $C(m,n)$, with integers $b > 0$, $m > 0$ and $n > 0$, as in figure 2.9. If $\triangle ABC$ were equilateral , then*

$$AB = BC = CA \implies b = \sqrt{(m-b)^2 + n^2} = \sqrt{m^2 + n^2}.$$

Squaring and expanding,

$$b^2 = m^2 - 2bm + b^2 + n^2 = m^2 + n^2.$$

From $BC = CA$ we deduce that

$$-2bm + b^2 = 0 \implies b(b-2m) \implies b = 2m,$$

as we are assuming that $b > 0$. Hence,

$$b^2 = m^2 + n^2 = \frac{b^2}{4} + n^2 \implies n = \frac{\sqrt{3}}{2}b.$$

Since we are assuming that $b \neq 0$, n cannot be an integer, since $\sqrt{3}$ is irrational. ◀

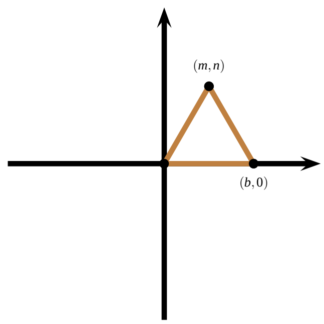

Figure 2.9: Example 80.

81 Theorem (Midpoint of a Line Segment) The point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ lies on the line joining $A(x_1,y_1)$ and $B(x_2,y_2)$, and it is equidistant from both points.

> **Proof:** *First observe that it is easy to find the midpoint of a vertical or horizontal line segment. The interval $[a;b]$ has length $b-a$. Hence, its midpoint is at $a + \frac{b-a}{2} = \frac{a+b}{2}$.*
>
> *Let (x,y) be the midpoint of the line segment joining $A(x_1,y_1)$ and $B(x_2,y_2)$. With $C(x_2,y_1)$, form the triangle $\triangle ABC$, right-angled at C. From (x,y), consider the projections of this point onto the line segments AC and BC. Notice that these projections are parallel to the legs of the triangle and so these projections pass through the midpoints of the legs. Since AC is a horizontal segment, its midpoint is at $M_B = (\frac{x_1+x_2}{2}, y_1)$. As BC is a horizontal segment, its midpoint is $M_A = (x_2, \frac{y_1+y_2}{2})$. The result is obtained on noting that (x,y) must have the same abscissa as M_B and the same ordinate as M_A.* ❑

In general, we have the following result.

82 Theorem (Joachimstal's Formula) The point P which divides the line segment AB, with $A(x_1,y_1)$ and $B(x_2,y_2)$, into two line segments in the ratio $m:n$ has coordinates

$$\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}\right).$$

> **Proof:** *The proof proceeds along the lines of Theorem 81. First we consider the interval $[a;b]$. Suppose that $a < x < b$ and that $\frac{x-a}{b-x} = \frac{m}{n}$. This gives $x = \frac{na+mb}{m+n}$.*
>
> *Form now $\triangle ABC$, right-angled at C. From P, consider the projection Q on AC and the projection R on BC. By Thales' Theorem, Q and R divide, respectively, AC and BC in the ratio $m:n$. By what was just demonstrated about intervals, the coordinates of Q are $\left(\frac{nx_1+mx_2}{m+n}, y_1\right)$ and the coordinates of R and $\left(x_2, \frac{ny_1+my_2}{m+n}\right)$, giving the result.* ❑

Homework

2.2.1 Problem Find $\mathbf{d}\langle(-2,-5),(4,-3)\rangle$.

2.2.2 Problem If a and b are real numbers, find the distance between the points (a,a) and (b,b).

2.2.3 Problem Find the distance between the points (a^2+a,b^2+b) and $(b+a,b+a)$.

2.2.4 Problem Demonstrate by direct calculation that

$$\mathbf{d}\langle(a,b),\left(\frac{a+c}{2},\frac{b+d}{2}\right)\rangle=\mathbf{d}\langle\left(\frac{a+c}{2},\frac{b+d}{2}\right),(c,d)\rangle.$$

2.2.5 Problem A car is located at point $A=(-x,0)$ and an identical car is located at point $(x,0)$. Starting at time $t=0$, the car at point A travels downwards at constant speed, at a rate of $a>0$ units per second and simultaneously, the car at point B travels upwards at constant speed, at a rate of $b>0$ units per second. How many units apart are these cars after $t>0$ seconds?

2.2.6 Problem Point C is at $\frac{3}{5}$ of the distance from $A(1,5)$ to $B(4,10)$ on the segment AB (and closer to B than to A). Find C.

2.2.7 Problem For which value of x is the point $(x,1)$ at distance 2 del from the point $(0,2)$?

2.2.8 Problem A bug starts at the point $(-1,-1)$ and wants to travel to the point $(2,1)$. In each quadrant, and on the axes, it moves with unit speed, except in quadrant II, where it moves with half the speed. Which route should the bug take in order to minimise its time? The answer is **not** a straight line from $(-1,-1)$ to $(2,1)$!

2.2.9 Problem Find the point equidistant from $(-1,0)$, $(1,0)$ and $(0,1/2)$.

2.2.10 Problem Find the coordinates of the point symmetric to (a,b) with respect to the point (b,a).

2.2.11 Problem Demonstrate that the diagonals of a rectangle are congruent.

2.2.12 Problem Prove that the diagonals of a parallelogram bisect each other..

2.2.13 Problem A fly starts at the origin and goes 1 unit up, $1/2$ unit right, $1/4$ unit down, $1/8$ unit left, $1/16$ unit up, etc., *ad infinitum.* In what coordinates does it end up?

2.2.14 Problem Find the coordinates of the point which is a quarter of the way from (a,b) to (b,a).

2.2.15 Problem Find the coordinates of the point symmetric to $(-a,b)$ with respect to: (i) the x-axis, (ii) the y-axis, (iii) the origin.

2.2.16 Problem (Minkowski's Inequality) Prove that if $(a,b),(c,d)\in\mathbb{R}^2$, then

$$\sqrt{(a+c)^2+(b+d)^2}\leq\sqrt{a^2+b^2}+\sqrt{c^2+d^2}.$$

Equality occurs if and only if $ad=bc$.

2.2.17 Problem Prove the following generalisation of Minkowski's Inequality. If $(a_k,b_k)\in(\mathbb{R}\setminus\{0\})^2, 1\leq k\leq n$, then

$$\sum_{k=1}^{n}\sqrt{a_k^2+b_k^2}\geq\sqrt{\left(\sum_{k=1}^{n}a_k\right)^2+\left(\sum_{k=1}^{n}b_k\right)^2}.$$

Equality occurs if and only if

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}.$$

2.2.18 Problem (AIME 1991) Let $P=\{a_1,a_2,\ldots,a_n\}$ be a collection of points with

$$0<a_1<a_2<\cdots<a_n<17.$$

Consider

$$S_n=\min_P\sum_{k=1}^{n}\sqrt{(2k-1)^2+a_k^2},$$

where the minimum runs over all such partitions P. Shew that exactly one of $S_2,S_3,\ldots,S_n,\ldots$ is an integer, and find which one it is.

2.3 Circles

The distance formula gives an algebraic way of describing points on the plane.

83 Theorem The equation of a circle with radius $R>0$ and centre (x_0,y_0) is

$$(x-x_0)^2+(y-y_0)^2=R^2. \tag{2.1}$$

This is called the *canonical equation* of the circle with centre $((x_0,y_0))$ and radius R.

Proof: *The point* (x,y) *belongs to the circle with radius* $R>0$ *if and only if its distance from the centre of the circle is R. This requires*

$$\begin{array}{rrcl}\iff & \mathbf{d}\langle (x,y),(x_0,y_0)\rangle & = & R\\ \iff & \sqrt{(x-x_0)^2+(y-y_0)^2} & = & R \quad ,\\ \iff & (x-x_0)^2+(y-y_0)^2 & = & R^2\end{array}$$

obtaining the result. See figure 2.10.❑

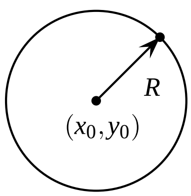

Figure 2.10: The circle.

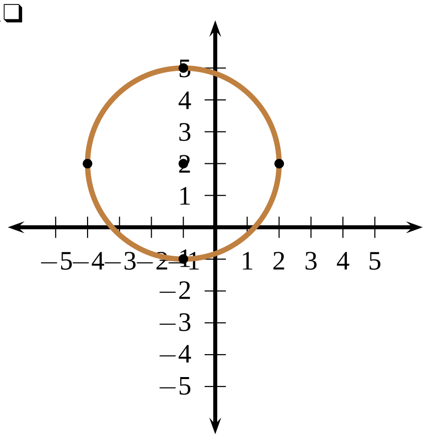

Figure 2.11: Example 84.

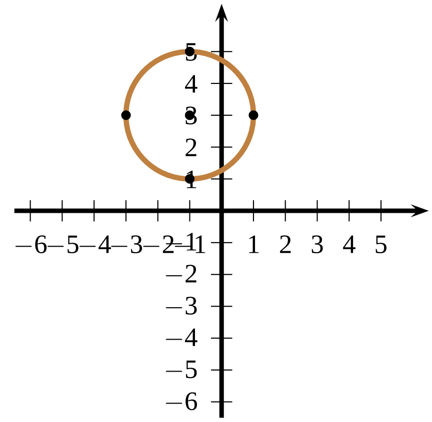

Figure 2.12: Example 85.

84 Example The equation of the circle with centre $(-1,2)$ and radius 3 is $(x+1)^2+(y-2)^2=9$. Observe that the points $(-1\pm 3,2)$ and $(-1,2\pm 3)$ are on the circle. Thus $(-4,2)$ is the left-most point on the circle, $(2,2)$ is the right-most, $(-1,-1)$ is the lower-most, and $(-1,5)$ is the upper-most. The circle is shewn in figure 2.11.

85 Example Trace the circle of equation

$$x^2+2x+y^2-6y=-6.$$

Solution: ▶ *Completing squares,*

$$x^2+2x+y^2-6y=-6 \implies x^2+2x+\mathbf{1}+y^2-6y+\mathbf{9}=-6+\mathbf{1}+\mathbf{9} \implies (x+1)^2+(y-3)^2=4,$$

from where we deduce that the centre of the circle is $(-1,3)$ *and the radius is* 2. *The point* $(-1+2,3)=(1,3)$ *lies on the circle, two units to the right of the centre. The point* $(-1-2,3)=(-3,3)$ *lies on the circle, two units to the left of the centre. The point* $(-1,3+2)=(-1,5)$ *lies on the circle, two unidades above the centre. The point* $(-1,3-2)=(-1,1)$ *lies on the circle, two unidades below the centre. See figure 2.12.* ◀

86 Example A diameter of a circle has endpoints $(-2,-1)$ and $(2,3)$. Find the equation of this circle and graph it.

Solution: ▶ *The centre of the circle lies on the midpoint of the diameter, thus the centre is* $\left(\frac{-2+2}{2},\frac{-1+3}{2}\right)=(0,1)$. *The equation of the circle is*

$$x^2+(y-1)^2=R^2.$$

To find the radius, we observe that $(2,3)$ *lies on the circle, thus*

$$2^2+(3-1)^2=R^2 \implies R=2\sqrt{2}.$$

The equation of the circle is finally

$$x^2+(y-1)^2=8.$$

Observe that the points $(0\pm 2\sqrt{2},1)$, $(0,1\pm 2\sqrt{2})$, *that is, the points* $(2\sqrt{2},1)$, $(-2\sqrt{2},1)$, $(0,1+2\sqrt{2})$, $(0,1-2\sqrt{2})$, $(-2,-1)$, *and* $(2,3)$ *all lie on the circle. The graph appears in figure 2.13.* ◀

87 Example Draw the plane region

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, \quad |x| \geq 1\}.$$

Solution: ► *Observe that* $|x| \geq 1 \iff x \geq 1$ *o* $x \leq -1$. *The region lies inside the circle with centre* $(0,0)$ *and radius* 2, *to the right of the vertical line* $x = 1$ *and to the left of the vertical line* $x = -1$. *See figure 2.14.*

◄

88 Example Find the equation of the circle passing through $(1,1)$, $(0,1)$ and $(1,2)$.

Solution: ► *Let* (h,k) *be the centre of the circle. Since the centre is equidistant from* $(1,1)$ *and* $(0,1)$, *we have,*

$$(h-1)^2 + (k-1)^2 = h^2 + (k-1)^2, \implies h^2 - 2h + 1 = h^2 \implies h = \frac{1}{2}.$$

Since he centre is equidistant from $(1,1)$ *and* $(1,2)$, *we have,*

$$(h-1)^2 + (k-1)^2 = (h-1)^2 + (k-2)^2 \implies k^2 - 2k + 1 = k^2 - 4k + 4 \implies k = \frac{3}{2}.$$

The centre of the circle is thus $(h,k) = (\frac{1}{2}, \frac{3}{2})$. *The radius of the circle is the distance from its centre to any point on the circle, say, to* $(0,1)$*:*

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2} - 1\right)^2} = \frac{\sqrt{2}}{2}.$$

The equation sought is finally

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{1}{2}.$$

See figure 2.15. ◄

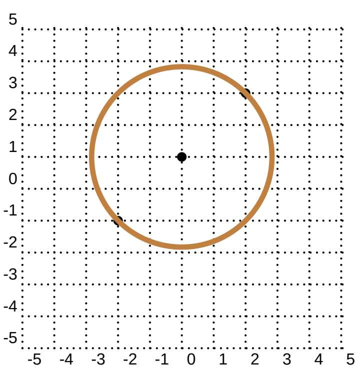

Figure 2.13: Example 2.13.

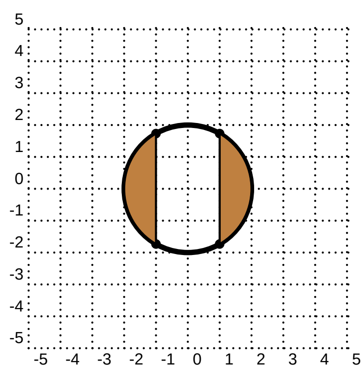

Figure 2.14: Example 87.

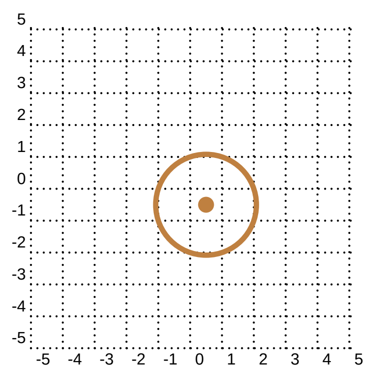

Figure 2.15: Example 88.

Homework

2.3.1 Problem Prove that the points $(4,2)$ and $(-2,-6)$ lie on the circle with centre at $(1,-2)$ and radius 5. Prove, moreover, that these two points are diametrically opposite.

2.3.2 Problem A diameter AB of a circle has endpoints $A = (1,2)$ and $B = (3,4)$. Find the equation of this circle.

2.3.3 Problem Find the equation of the circle with centre at $(-1,1)$ and passing through $(1,2)$.

2.3.4 Problem Rewrite the following circle equations in canonical form and find their centres C and their radius R. Draw the circles. Also, find at least four points belonging to each circle.

1. $x^2+y^2-2y=35$,
2. $x^2+4x+y^2-2y=20$,
3. $x^2+4x+y^2-2y=5$,
4. $2x^2-8x+2y^2=16$,
5. $4x^2+4x+\frac{15}{2}+4y^2-12y=0$
6. $3x^2+2x\sqrt{3}+5+3y^2-6y\sqrt{3}=0$

2.3.5 Problem Let

$$\begin{aligned} R_1 &= \{(x,y)\in\mathbb{R}^2 | x^2+y^2\leq 9\},\\ R_2 &= \{(x,y)\in\mathbb{R}^2 | (x+2)^2+y^2\leq 1\},\\ R_3 &= \{(x,y)\in\mathbb{R}^2 | (x-2)^2+y^2\leq 1\},\\ R_4 &= \{(x,y)\in\mathbb{R}^2 | x^2+(y+1)^2\leq 1\},\\ R_5 &= \{(x,y)\in\mathbb{R}^2 | |x|\leq 3, |y|\leq 3\},\\ R_6 &= \{(x,y)\in\mathbb{R}^2 | |x|\geq 2, |y|\geq 2\}.\end{aligned}$$

Sketch the following regions.

1. $R_1\setminus(R_2\cup R_3\cup R_4)$.
2. $R_5\setminus R_1$
3. $R_1\setminus R_6$
4. $R_2\cup R_3\cup R_6$

2.3.6 Problem Find the equation of the circle passing through $(-1,2)$ and centre at $(1,3)$.

2.3.7 Problem Find the canonical equation of the circle passing through $(-1,1)$, $(1,-2)$, and $(0,2)$.

2.3.8 Problem Let a,b,c be real numbers with $a^2>4b$. Construct a circle with diameter at the points $(1,0)$ and $(-a,b)$. Shew that the intersection of this circle with the x-axis are the roots of the equation $x^2+ax+b=0$. Why must we impose $a^2>4b$?

2.3.9 Problem Draw

$$(x^2+y^2-100)((x-4)^2+y^2-4)((x+4)^2+y^2-4)(x^2+(y+4)^2-4)=0.$$

2.4 Semicircles

Given a circle of centre (a,b) and radius $R>0$, its canonical equation is

$$(x-a)^2+(y-b)^2=R^2.$$

Solving for y we gather

$$(y-b)^2=R^2-(x-a)^2 \implies y=b\pm\sqrt{R^2-(x-a)^2}.$$

If we took the $+$ sign on the square root, then the values of y will lie above the line $y=b$, and hence $y=b+\sqrt{R^2-(x-a)^2}$ is the equation of the upper semicircle with centre at (a,b) and radius $R>0$. Also, $y=b-\sqrt{R^2-(x-a)^2}$ is the equation of the lower semicircle.

In a similar fashion, solving for x we obtain,

$$(x-a)^2=R^2-(y-b)^2 \implies x=a\pm\sqrt{R^2-(y-b)^2}.$$

Taking the $+$ sign on the square root, the values of x will lie to the right of the line $x=a$, and hence $x=a+\sqrt{R^2-(y-b)^2}$ is the equation of the right semicircle with centre at (a,b) and radius $R>0$. Similarly, $x=a-\sqrt{R^2-(y-b)^2}$ is the equation of the left semicircle.

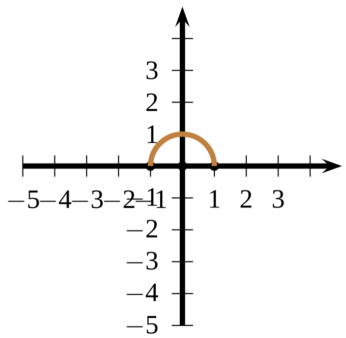

Figure 2.16: Example 89.

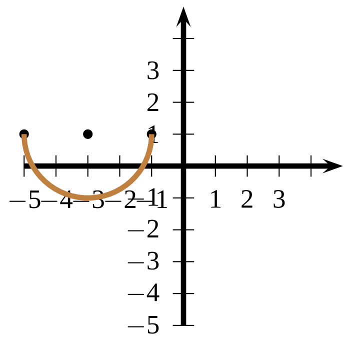

Figure 2.17: Example 90.

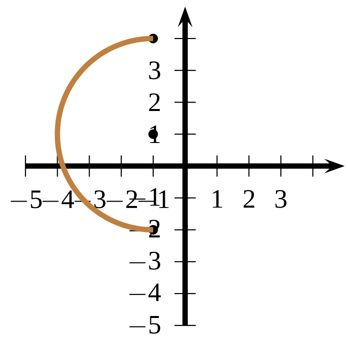

Figure 2.18: Example 91.

89 Example Figure 2.16 shews the upper semicircle $y = \sqrt{1-x^2}$.

90 Example Draw the semicircle of equation $y = 1 - \sqrt{-x^2-6x-5}$.

> **Solution:** ► *Since the square root has a minus sign, the semicircle will be a lower semicircle, lying below the line $y = 1$. We must find the centre and the radius of the circle . For this, let us complete the equation of the circle by squaring and rearranging. This leads to*
>
> $$\begin{aligned} y = 1 - \sqrt{-x^2-6x-5} \quad & \Longrightarrow \quad y - 1 = -\sqrt{-x^2-6x-5} \\ & \Longrightarrow \quad (y-1)^2 = -x^2-6x-5 \\ & \Longrightarrow \quad x^2+6x+\mathbf{9}+(y-1)^2 = -5+\mathbf{9} \\ & \Longrightarrow \quad (x+3)^2+(y-1)^2 = 4, \end{aligned}$$
>
> *whence the semicircle has centre at $(-3,1)$ and radius 2. Its graph appears in figure 2.17.* ◄

91 Example Find the equation of the semicircle in figure 2.18.

> **Solution:** ► *The semicircle has centre at $(-1,1)$ and radius 3. The full circle would have equation*
>
> $$(x+1)^2+(y-1)^2 = 9.$$
>
> *Since this is a left semicircle, we must solve for x and take the minus $-$ on the square root:*
>
> $$(x+1)^2+(y-1)^2 = 9 \implies (x+1)^2 = 9-(y-1)^2 \implies x+1 = -\sqrt{9-(y-1)^2} \implies x = -1-\sqrt{9-(y-1)^2},$$
>
> *whence the equation sought is $x = -1-\sqrt{9-(y-1)^2}$.* ◄

Homework

2.4.1 Problem Sketch the following curves.

1. $y = \sqrt{16-x^2}$
2. $x = -\sqrt{16-y^2}$
3. $x = -\sqrt{12-4y-y^2}$
4. $x = -5-\sqrt{12+4y-y^2}$

2.4.2 Problem Draw

$$(x^2+y^2-100)(y-\sqrt{4-(x+4)^2})(y-\sqrt{4-(x-4)^2})(y+4+\sqrt{4-x^2}) = 0.$$

2.5 Lines

In the previous sections we saw the link *Algebra to Geometry* by giving the equation of a circle and producing its graph, and conversely, the link *Geometry to Algebra* by starting with the graph of a circle and finding its equation. This section will continue establishing these links, but our focus now will be on lines.

We have already seen equations of vertical and horizontal lines. We give their definition again for the sake of completeness.

92 Definition Let a and b be real number constants. A *vertical line* on the plane is a set of the form

$$\{(x,y) \in \mathbb{R}^2 : x = a\}.$$

Similarly, a *horizontal line* on the plane is a set of the form

$$\{(x,y) \in \mathbb{R}^2 : y = b\}.$$

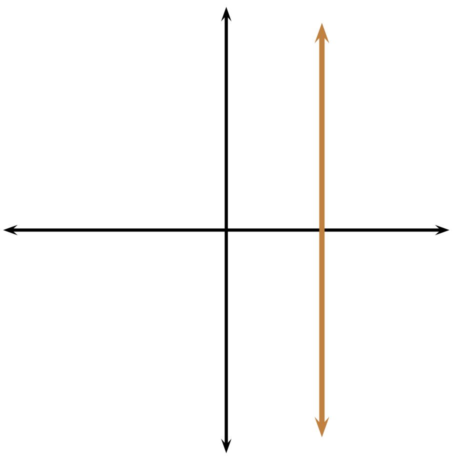

Figure 2.19: A vertical line.

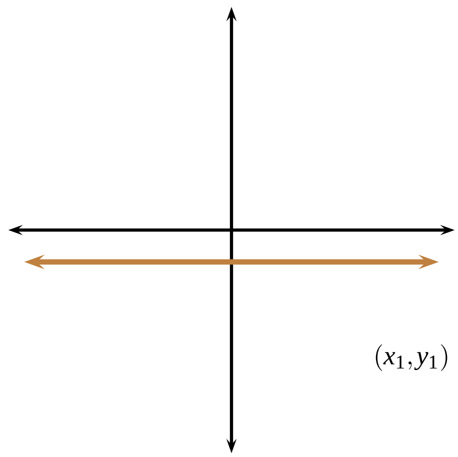

Figure 2.20: A horizontal line.

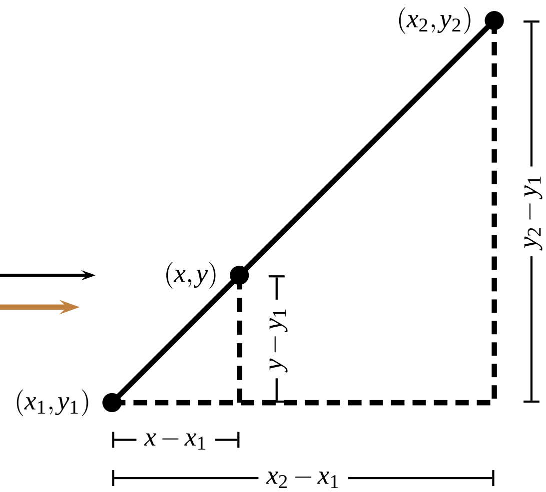

Figure 2.21: Theorem 93.

93 Theorem The equation of any non-vertical line on the plane can be written in the form $y = mx + k$, where m and k are real number constants. Conversely, any equation of the form $y = ax + b$, where a, b are fixed real numbers has as a line as a graph.

> **Proof:** *If the line is parallel to the x-axis, that is, if it is horizontal, then it is of the form $y = b$, where b is a constant and so we may take $m = 0$ and $k = b$. Consider now a line non-parallel to any of the axes, as in figure 2.21, and let (x,y), (x_1,y_1), (x_2,y_2) be three given points on the line. By similar triangles we have*
>
> $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1},$$
>
> *which, upon rearrangement, gives*
>
> $$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x - x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1,$$
>
> *and so we may take*
>
> $$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad k = -x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1.$$
>
> *Conversely, consider real numbers $x_1 < x_2 < x_3$, and let $P = (x_1, ax_1 + b)$, $Q = (x_2, ax_2 + b)$, and $R = (x_3, ax_3 + b)$ be on the graph of the equation $y = ax + b$. We will shew that*
>
> $$\mathbf{d}\langle P,Q\rangle + \mathbf{d}\langle Q,R\rangle = \mathbf{d}\langle P,R\rangle.$$
>
> *Since the points P, Q, R are arbitrary, this means that any three points on the graph of the equation $y = ax + b$ are collinear, and so this graph is a line. Then*
>
> $$\mathbf{d}\langle P,Q\rangle = \sqrt{(x_2 - x_1)^2 + (ax_2 - ax_1)^2} = |x_2 - x_1|\sqrt{1 + a^2} = (x_2 - x_1)\sqrt{1 + a^2},$$
>
> $$\mathbf{d}\langle Q,R\rangle = \sqrt{(x_3 - x_2)^2 + (ax_3 - ax_2)^2} = |x_3 - x_2|\sqrt{1 + a^2} = (x_3 - x_2)\sqrt{1 + a^2},$$

$$\mathbf{d}\langle P,Q\rangle = \sqrt{(x_3-x_1)^2+(ax_3-ax_1)^2} = |x_3-x_1|\sqrt{1+a^2} = (x_3-x_1)\sqrt{1+a^2},$$

from where

$$\mathbf{d}\langle P,Q\rangle + \mathbf{d}\langle Q,R\rangle = \mathbf{d}\langle P,R\rangle$$

follows. This means that the points P,Q, *and* R *lie on a straight line, which finishes the proof of the theorem.* ❑

94 Definition The quantity $m = \dfrac{y_2-y_1}{x_2-x_1}$ in Theorem 93 is the *slope or gradient of the line passing through* (x_1,y_1) *and* (x_2,y_2). Since $y = m(0)+k$, the point $(0,k)$ is the *y-intercept of the line joining* (x_1,y_1) *and* (x_2,y_2). Figures 2.22 through 2.25 shew how the various inclinations change with the sign of m.

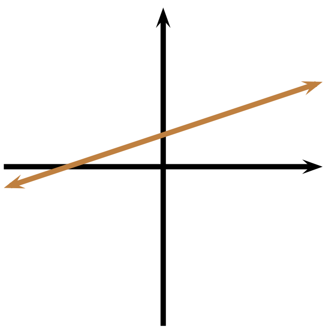

Figure 2.22: $m>0$

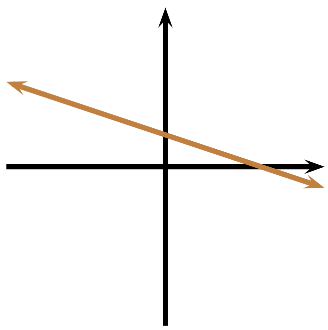

Figure 2.23: $m<0$

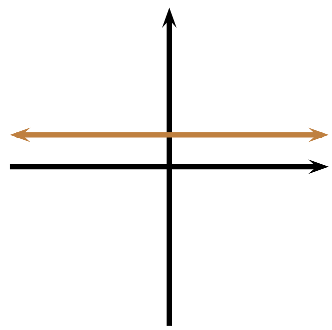

Figure 2.24: $m=0$

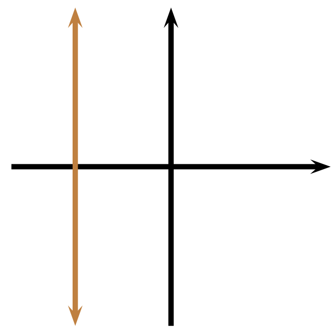

Figure 2.25: $m=\infty$

95 Example By Theorem 93, the equation $y = x$ represents a line with slope 1 and passing through the origin. Since $y = x$, the line makes a 45° angle with the x-axis, and bisects quadrants I and III. See figure 2.26

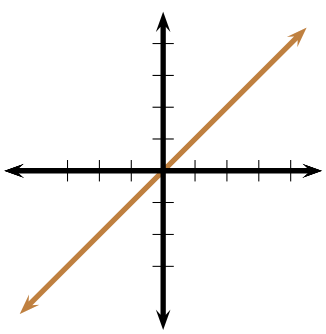

Figure 2.26: Example 95.

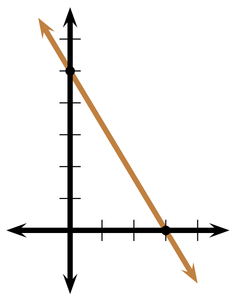

Figure 2.27: Example 96.

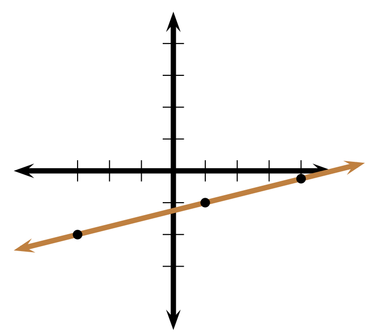

Figure 2.28: Example 97.

96 Example A line passes through $(-3,10)$ and $(6,-5)$. Find its equation and draw it.

Solution: ▶ *The equation is of the form* $y = mx+k$. *We must find the slope and the y-intercept. To find* m *we compute the ratio*

$$m = \frac{10-(-5)}{-3-6} = -\frac{5}{3}.$$

Thus the equation is of the form $y = -\frac{5}{3}x+k$ *and we must now determine* k. *To do so, we substitute either point, say the first, into* $y = -\frac{5}{3}x+k$ *obtaining* $10 = -\frac{5}{3}(-3)+k$, *whence* $k = 5$. *The equation sought is thus* $y = -\frac{5}{3}x+5$. *To draw the graph, first locate the y-intercept (at* $(0,5)$*). Since the slope is* $-\frac{5}{3}$, *move five units down (to* $(0,0)$*) and three to the right (to* $(3,0)$*). Connect now the points* $(0,5)$ *and* $(3,0)$. *The graph appears in figure* 2.27. ◀

97 Example Three points $(4,u), (1,-1)$ and $(-3,-2)$ lie on the same line. Find u.

> **Solution:** ► *Since the points lie on the same line, any choice of pairs of points used to compute the gradient must yield the same quantity. Therefore*
>
> $$\frac{u-(-1)}{4-1} = \frac{-1-(-2)}{1-(-3)}$$
>
> *which simplifies to the equation*
>
> $$\frac{u+1}{3} = \frac{1}{4}.$$
>
> *Solving for u we obtain $u = -\frac{1}{4}$.* ◄

Homework

2.5.1 Problem Assuming that the equations for the lines l_1, l_2, l_3, and l_4 in figure 2.29 below can be written in the form $y = mx + b$ for suitable real numbers m and b, determine which line has the largest value of m and which line has the largest value of b.

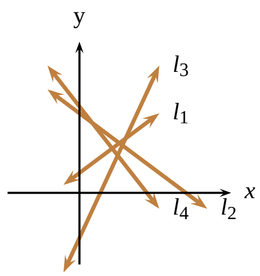

Figure 2.29: Problem 2.5.1.

2.5.2 Problem (AHSME 1994) Consider the L-shaped region in the plane, bounded by horizontal and vertical segments with vertices at $(0,0), (0,3), (3,3), (3,1), (5,1)$ and $(5,0)$. Find the gradient of the line that passes through the origin and divides this area exactly in half.

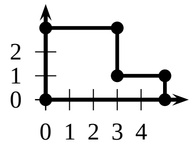

Figure 2.30: Problem 2.5.2.

2.5.3 Problem What is the slope of the line with equation $\frac{x}{a} + \frac{y}{b} = 1$?

2.5.4 Problem If the point $(a, -a)$ lies on the line with equation $-2x + 3y = 30$, find the value of a.

2.5.5 Problem Find the equation of the straight line joining $(3,1)$ and $(-5,-1)$.

2.5.6 Problem Let $(a,b) \in \mathbb{R}^2$. Find the equation of the straight line joining (a,b) and (b,a).

2.5.7 Problem Find the equation of the line that passes through (a, a^2) and (b, b^2).

2.5.8 Problem The points $(1,m), (2,4)$ lie on a line with gradient m. Find m.

2.5.9 Problem Consider the following regions on the plane.

$$R_1 = \{(x,y) \in \mathbb{R}^2 | y \leq 1 - x\},$$
$$R_2 = \{(x,y) \in \mathbb{R}^2 | y \geq x + 2\},$$
$$R_3 = \{(x,y) \in \mathbb{R}^2 | y \leq 1 + x\}.$$

Sketch the following regions.

1. $R_1 \setminus R_2$
2. $R_2 \setminus R_1$
3. $R_1 \cap R_2 \cap R_3$
4. $R_2 \setminus (R_1 \cup R_2)$

2.5.10 Problem In figure 2.31, point M has coordinates $(2,2)$, points A, S are on the x-axis, point B is on the y-axis $\triangle SMA$ is isosceles at M, and the line segment SM has slope 2. Find the coordinates of points A, B, S.

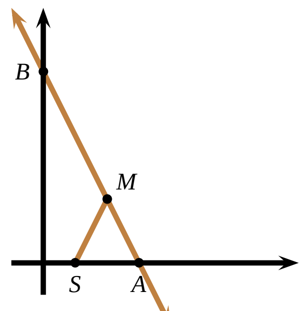

Figure 2.31: Problem 2.5.10.

2.5.11 Problem Which points on the line with equation $y = 6 - 2x$ are equidistant from the axes?

2.5.12 Problem A vertical line divides the triangle with vertices $(0,0), (1,1)$ and $(9,1)$ in the plane into two regions of equal area. Find the equation of this vertical line.

2.5.13 Problem Draw
$$(x^2-1)(y^2-1)(x^2-y^2)=0.$$

2.6 Parallel and Perpendicular Lines

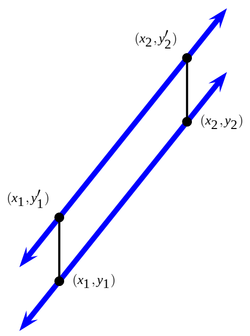

Figure 2.32: Theorem 98.

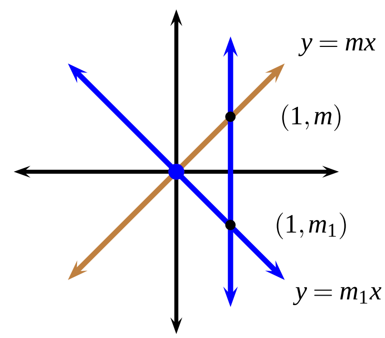

Figure 2.33: Theorem 100.

98 Theorem Two lines are parallel if and only if they have the same slope.

Proof: *Suppose the the lines L and L′ are parallel, and that the points $A(x_1,y_1)$ y $B(x_2,y_2)$ lie on L and that the points $A'(x_1,y_1')$ and $B'(x_2,y_2')$ lie on L′. Observe tha t $ABB'A'$ is a parallelogram, and hence, $y_2-y_1=y_2'-y_1'$, which gives*
$$\frac{y_2-y_1}{x_2-x_1}=\frac{y_2'-y_1'}{x_2-x_1},$$
demonstrating that the slopes of L and L′ are equal.

Assume now that L and L′ have the same slope. The
$$\frac{y_2-y_1}{x_2-x_1}=\frac{y_2'-y_1'}{x_2-x_1}\implies y_2-y_1=y_2'-y_1'.$$
Then the sides of AA' and BB' of the quadrilateral $ABB'A'$ are congruent. As these sides are also parallel, since they are on the verticals $x=x_1$ and $x=x_2$, we deduce that $ABB'A'$ is a parallelogram, demonstrating that L and L′ are parallel. ❑

99 Example Find the equation of the line passing through $(4,0)$ and parallel to the line joining $(-1,2)$ and $(2,-4)$.

Solution: ► *First we compute the slope of the line joining $(-1,2)$ and $(2,-4)$:*
$$m=\frac{2-(-4)}{-1-2}=-2.$$
The line we seek is of the form $y=-2x+k$. We now compute the y-intercept, using the fact that the line must pass through $(4,0)$. This entails solving $0=-2(4)+k$, whence $k=8$. The equation sought is finally $y=-2x+8$. ◄

100 Theorem Let $y=mx+k$ be a line non-parallel to the axes. If the line $y=m_1x+k_1$ is perpendicular to $y=mx+k$ then $m_1=-\dfrac{1}{m}$. Conversely, if $mm_1=-1$, then the lines with equations $y=mx+k$ and $y=m_1x+k_1$ are perpendicular.

Proof: *Refer to figure 2.33. Since we may translate lines without affecting the angle between them, we assume without loss of generality that both $y = mx + k$ and $y = m_1x + k_1$ pass through the origin, giving thus $k = k_1 = 0$. Now, the line $y = mx$ meets the vertical line $x = 1$ at $(1, m)$ and the line $y = m_1x$ meets this same vertical line at $(1, m_1)$ (see figure 2.33). By the Pythagorean Theorem*

$$(m - m_1)^2 = (1 + m^2) + (1 + m_1^2) \implies m^2 - 2mm_1 + m_1^2 = 2 + m^2 + m_1^2 \implies mm_1 = -1,$$

which proves the assertion. The converse is obtained by retracing the steps and using the converse to the Pythagorean Theorem. ❑

101 Example Find the equation of the line passing through $(4, 0)$ and perpendicular to the line joining $(-1, 2)$ and $(2, -4)$.

Solution: ▶ *The slope of the line joining $(-1, 2)$ and $(2, -4)$ is -2. The slope of any line perpendicular to it*

$$m_1 = -\frac{1}{m} = \frac{1}{2}.$$

The equation sought has the form $y = \frac{x}{2} + k$. We find the y-intercept by solving $0 = \frac{4}{2} + k$, whence $k = -2$. The equation of the perpendicular line is thus $y = \frac{x}{2} - 2$. ◀

102 Example For a given real number t, associate the straight line L_t with the equation

$$L_t: \quad (4 - t)y = (t + 2)x + 6t.$$

1. Determine t so that the point $(1, 2)$ lies on the line L_t and find the equation of this line.
2. Determine t so that the L_t be parallel to the x-axis and determine the equation of the resulting line.
3. Determine t so that the L_t be parallel to the y-axis and determine the equation of the resulting line.
4. Determine t so that the L_t be parallel to the line $-5y = 3x - 1$.
5. Determine t so that the L_t be perpendicular to the line $-5y = 3x - 1$.
6. Is there a point (a, b) belonging to every line L_t regardless of the value of t?

Solution: ▶

1. *If the point $(1, 2)$ lies on the line L_t then we have*

$$(4 - t)(2) = (t + 2)(1) + 6t \implies t = \frac{2}{3}.$$

The line sought is thus

$$L_{2/3}: \quad (4 - \frac{2}{3})y = (\frac{2}{3} + 2)x + 6\left(\frac{2}{3}\right)$$

or $y = \frac{4}{5}x + \frac{6}{5}$.

2. *We need $t + 2 = 0 \implies t = -2$. In this case*

$$(4 - (-2))y = -12 \implies y = -2.$$

3. *We need $4 - t = 0 \implies t = 4$. In this case*

$$0 = (4 + 2)x + 24 \implies x = -4.$$

4. *The slope of* L_t *is*
$$\frac{t+2}{4-t},$$
and the slope of the line $-5y = 3x-1$ *is* $-\frac{3}{5}$. *Therefore we need*
$$\frac{t+2}{4-t} = -\frac{3}{5} \implies -3(4-t) = 5(t+2) \implies t = -11.$$

5. *In this case we need*
$$\frac{t+2}{4-t} = \frac{5}{3} \implies 5(4-t) = 3(t+2) \implies t = \frac{7}{4}.$$

6. *Yes. From above, the obvious candidate is* $(-4,-2)$. *To verify this observe that*
$$(4-t)(-2) = (t+2)(-4)+6t,$$
regardless of the value of t.

◄

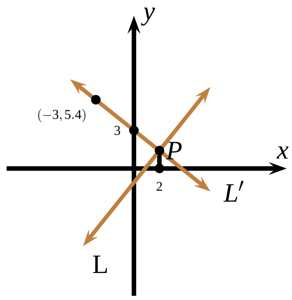

Figure 2.34: Example 103.

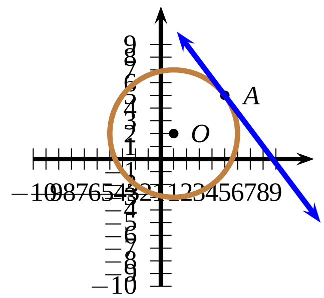

Figure 2.35: Example 104.

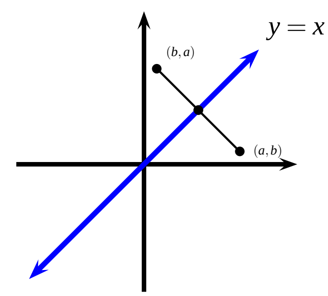

Figure 2.36: Theorem 107.

103 Example In figure 2.34, the straight lines L y L' are perpendicular and meet at the point P.

1. Find the equation of L'.
2. Find the coordinates of P.
3. Find the equation of the line L.

Solution: ►

1. *Notice that* L' *passes through* $(-3,5.4)$ *and through* $(0,3)$, *hence it must have slope*
$$\frac{5.4-3}{-3-0} = -0.8.$$
The equation of L' *has the form* $y = -0.8x+k$. *Since* L' *passes through* $(0,3)$, *we deduce that* L' *has equation* $y = -0.8x+3$.
2. *Since* P *if of the form* $(2,y)$ *and since it lies on* L', *we deduce that* $y = -0.8(2)+3 = 1.4$.
3. L *has slope* $-\frac{1}{-0.8} = 1.25$. *This means that* L *has equation of the form* $y = 1.25x+k$. *Since* $P(2,1.4)$ *lies on* L, *we must have* $1.4 = 1.25(2)+k \implies k = -1.1$. *We deduce that* L *has equation* $y = 1.25x-1.1$.

◄

104 Example Consider the circle $\mathscr{C}$ of centre $O(1,2)$ and passing through $A(5,5)$, as in figure D.183.

1. Find the equation of $\mathscr{C}$.
2. Find all the possible values of a for which the point $(2,a)$ lies on the circle $\mathscr{C}$.
3. Find the equation of the line L tangent to $\mathscr{C}$ at A.

Solution: ►

1. *Let $R > 0$ be the radius of the circle . Then equation of the circle has the form*
$$(x-1)^2 + (y-2)^2 = R^2.$$
Since $A(5,5)$ lies on the circle,
$$(5-1)^2 + (5-2)^2 = R^2 \implies 16 + 9 = R^2 \implies 25 = R^2,$$
whence the equation sought for $\mathscr{C}$ is
$$(x-1)^2 + (y-2)^2 = 25.$$
2. *If the point $(2,a)$ lies on $\mathscr{C}$, we will have*
$$(2-1)^2 + (a-2)^2 = 25 \implies 1 + (a-2)^2 = 25 \implies (a-2)^2 = 24 \implies a-2 = \pm\sqrt{24} \implies a = 2 \pm \sqrt{24} = 2 \pm 2\sqrt{6}.$$
3. *L is perpendicular to the line joining $(1,2)$ and $(5,5)$. As this last line has slope*
$$\frac{5-2}{5-1} = \frac{3}{4},$$
the line L will have slope $-\frac{4}{3}$. Thus L has equation of the form
$$y = -\frac{4}{3}x + k.$$
As $(5,5)$ lies on the line,
$$5 = -\frac{4}{3} \cdot 5 + k \implies 5 + \frac{20}{3} = k \implies k = \frac{35}{3},$$
from where we gather that L has equation $y = -\frac{4}{3}x + \frac{35}{3}$.

◄

We will now demonstrate two results that will be needed later.

105 Theorem (Distance from a Point to a Line) Let $L : y = mx + k$ be a line on the plane and let $P = (x_0, y_0)$ be a point on the plane, not on L. The distance $\mathbf{d}\langle L, P\rangle$ from L to P is given by
$$\frac{|x_0 m + k - y_0|}{\sqrt{1+m^2}}.$$

Proof: *If the line had infinite slope, then L would be vertical, and of equation $x = c$, for some constant c, and then clearly,*
$$\mathbf{d}\langle L, P\rangle = |x_0 - c|.$$
If $m = 0$, then L would be horizontal, and then clearly
$$\mathbf{d}\langle L, P\rangle = |y_0 - k|,$$

agreeing with the theorem. Suppose now that $m \neq 0$. Refer to figure 2.37. The line L has slope m and all perpendicular lines to L must have slope $-\frac{1}{m}$. The distance from P to L is the length of the line segment joining P with the point of intersection (x_1, y_1) of the line L' perpendicular to L and passing through P. Now, it is easy to see that L' has equation

$$L' : y = -\frac{1}{m}x + y_0 + \frac{x_0}{m},$$

from where L and L' intersect at

$$x_1 = \frac{y_0 m + x_0 - bm}{1+m^2}, \quad y_1 = \frac{y_0 m^2 + x_0 m + k}{1+m^2}.$$

This gives

$$\begin{aligned}
\mathbf{d}\langle L, P\rangle &= \mathbf{d}\langle (x_0, y_0), (x_1, y_1)\rangle \\
&= \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} \\
&= \sqrt{\left(x_0 - \frac{y_0 m + x_0 - km}{1+m^2}\right)^2 + \left(y_0 - \frac{y_0 m^2 + x_0 m + k}{1+m^2}\right)^2} \\
&= \frac{\sqrt{(x_0 m^2 - y_0 m + km)^2 + (y_0 - x_0 m - k)^2}}{1+m^2} \\
&= \frac{\sqrt{(m^2+1)(x_0 m - y_0 + k)^2}}{1+m^2} \\
&= \frac{|x_0 m - y_0 + k|}{\sqrt{1+m^2}},
\end{aligned}$$

proving the theorem.

Aliter: *A "proof without words" can be obtained by considering the similar right triangles in figure 2.38.* ❑

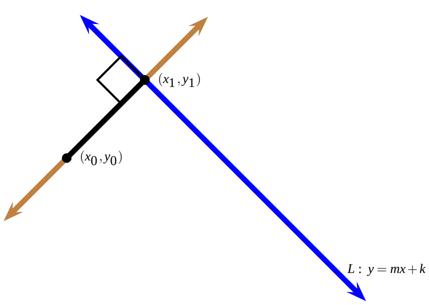

Figure 2.37: Theorem 105.

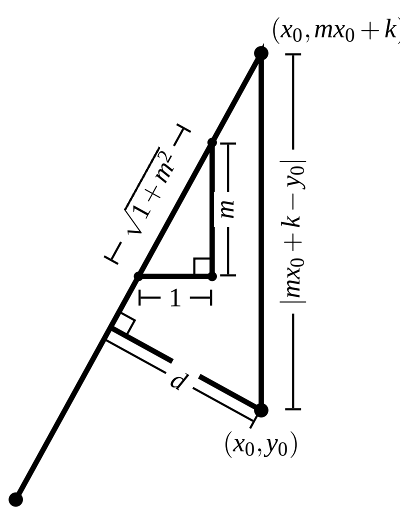

Figure 2.38: Theorem 105.

106 Example Find the distance between the line $L : 2x - 3y = 1$ and the point $(-1, 1)$.

Solution: ► *The equation of the line L can be rewritten in the form $L: y = \frac{2}{3}x - \frac{1}{3}$. Using Theorem 105, we have*

$$\mathbf{d}\langle L, P\rangle = \frac{|-\frac{2}{3} - 1 - \frac{1}{3}|}{\sqrt{1 + (\frac{2}{3})^2}} = \frac{6\sqrt{13}}{13}.$$

◄

107 Theorem The point (b, a) is symmetric to the point (a, b) with respect to the line $y = x$.

Proof: *The line joining (b, a) to (a, b) has equation $y = -x + a + b$. This line is perpendicular to the line $y = x$ and intersects it when*

$$x = -x + a + b \implies x = \frac{a+b}{2}.$$

Then, since $y = x = \dfrac{a+b}{2}$, the point of intersection is $(\frac{a+b}{2}, \frac{a+b}{2})$. But this point is the midpoint of the line segment joining (a, b) to (b, a), which means that both (a, b) and (b, a) are equidistant from the line $y = x$, establishing the result. See figure 2.36. ❑

Homework

2.6.1 Problem Find the equation of the straight line parallel to the line $8x - 2y = 6$ and passing through $(5, 6)$.

2.6.2 Problem Let $(a, b) \in (\mathbb{R} \setminus \{0\})^2$. Find the equation of the line passing through (a, b) and parallel to the line $\frac{x}{a} - \frac{y}{b} = 1$.

2.6.3 Problem Find the equation of the straight line normal to the line $8x - 2y = 6$ and passing through $(5, 6)$.

2.6.4 Problem Let a, b be strictly positive real numbers. Find the equation of the line passing through (a, b) and perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$.

2.6.5 Problem Find the equation of the line passing through $(12, 0)$ and parallel to the line joining $(1, 2)$ and $(-3, -1)$.

2.6.6 Problem Find the equation of the line passing through $(12, 0)$ and normal to the line joining $(1, 2)$ and $(-3, -1)$.

2.6.7 Problem Find the equation of the straight line tangent to the circle $x^2 + y^2 = 1$ at the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

2.6.8 Problem Consider the line L passing through (a, a^2) and (b, b^2). Find the equations of the lines L_1 parallel to L and L_2 normal to L, if L_1 and L_2 must pass through $(1, 1)$.

2.6.9 Problem For any real number t, associate the straight line L_t having equation

$$(2t - 1)x + (3 - t)y - 7t + 6 = 0.$$

In each of the following cases, find an t satisfying the stated conditions.

1. L_t passes through $(1, 1)$.
2. L_t passes through the origin $(0, 0)$.
3. L_t is parallel to the x-axis.
4. L_t is parallel to the y-axis.
5. L_t is parallel to the line of equation $3x - 2y - 6 = 0$.
6. L_t is normal to the line of equation $y = 4x - 5$.
7. L_t has gradient -2.
8. Is there a point (x_0, y_0) belonging to L_t no matter which real number t be chosen?

2.6.10 Problem For any real number t, associate the straight line L_t having equation

$$(t - 2)x + (t + 3)y + 10t - 5 = 0.$$

In each of the following cases, find an t and the resulting line satisfying the stated conditions.

1. L_t passes through $(-2, 3)$.
2. L_t is parallel to the x-axis.
3. L_t is parallel to the y-axis.
4. L_t is parallel to the line of equation $x - 2y - 6 = 0$.
5. L_t is normal to the line of equation $y = -\frac{1}{4}x - 5$.
6. Is there a point (x_0, y_0) belonging to L_t no matter which real number t be chosen?

2.6.11 Problem Shew that the four points $A = (-2, 0)$, $B = (4, -2)$, $C = (5, 1)$, and $D = (-1, 3)$ form the vertices of a rectangle.

2.6.12 Problem Find the distance from the point $(1, 1)$ to the line $y = -x$.

2.6.13 Problem Let $a \in \mathbb{R}$. Find the distance from the point $(a, 0)$ to the line $L: y = ax + 1$.

2.6.14 Problem Find the equation of the circle with centre at $(3, 4)$ and tangent to the line $x - 2y + 3 = 0$.

2.6.15 Problem $\triangle ABC$ has vertices at $A(a,0)$, $B(b,0)$ and $C(0,c)$, where $a<0<b$. Demonstrate, using coordinates, that the medians of $\triangle ABC$ are concurrent at the point $\left(\frac{a+b}{3},\frac{c}{3}\right)$. The point of concurrence is called the *barycentre* or *centroid* of the triangle.

2.6.16 Problem $\triangle ABC$ has vertices at $A(a,0)$, $B(b,0)$ and $C(0,c)$, where $a<0<b$, $c\neq 0$. Demonstrate, using coordinates, that the altitudes of $\triangle ABC$ are concurrent at the point $\left(0,-\frac{ab}{c}\right)$. The point of concurrence is called the *orthocentre* of the triangle.

2.6.17 Problem $\triangle ABC$ has vertices at $A(a,0)$, $B(b,0)$ y $C(0,c)$, where $a<0<b$. Demonstrate, using coordinates, that the perpendicular bisectors of $\triangle ABC$ are concurrent at the point $\left(\frac{a+b}{2},\frac{ab+c^2}{2c}\right)$. The point of concurrence is called the *circumcentre* of the triangle.

2.6.18 Problem Demonstrate that the diagonals of a square are mutually perpendicular.

2.7 Linear Absolute Value Curves

In this section we will use the sign diagram methods of section 1.5 in order to decompose certain absolute value curves as the union of lines.

108 Example Since

$$|x|=\begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } x<0\end{cases}$$

the graph of the curve $y=|x|$ is that of the line $y=-x$ for $x<0$ and that of the line $y=x$ when $x\geq 0$. The graph can be seen in figure F.4.

109 Example Draw the graph of the curve with equation $y=|2x-1|$.

Solution: ▶ *Recall that either $|2x-1|=2x-1$ or that $|2x-1|=-(2x-1)$, depending on the sign of $2x-1$. If $2x-1\geq 0$ then $x\geq\frac{1}{2}$ and so we have $y=2x-1$. This means that for $x\geq\frac{1}{2}$, we will draw the graph of the line $y=2x-1$. If $2x-1<0$ then $x<\frac{1}{2}$ and so we have $y=-(2x-1)=1-2x$. This means that for $x<\frac{1}{2}$, we will draw the graph of the line $y=1-2x$. The desired graph is the union of these two graphs and appears in figure 2.40.* ◀

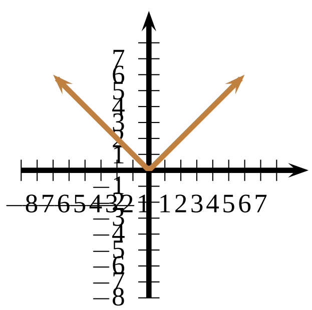

Figure 2.39: $y=|x|$.

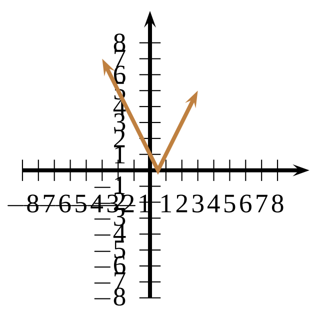

Figure 2.40: Example 109.

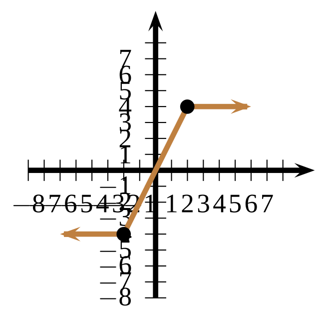

Figure 2.41: Example 110.

110 Example Consider the equation $y=|x+2|-|x-2|$. The terms in absolute values vanish when $x=-2$ or $x=-2$. If $x\leq -2$ then

$$|x+2|-|x-2|=(-x-2)-(-x+2)=-4.$$

For $-2\leq x\leq 2$, we have

$$|x+2|-|x-2|=(x+2)-(-x+2)=2x.$$

For $x \geq 2$, we have

$$|x+2|-|x-2| = (x+2)-(x-2) = 4.$$

Then,

$$y = |x+2|-|x-2| = \begin{cases} -4 & \text{if } x \leq -2, \\ 2x & \text{if } -2 \leq x \leq +2, \\ +4 & \text{if } x \geq +2, \end{cases}$$

The graph is the union of three lines (or rather, two rays and a line segment), and can be see in figure F.5.

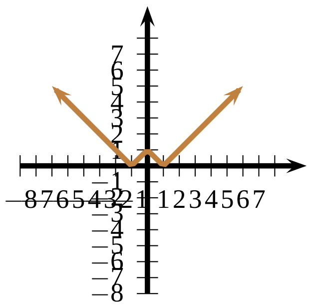

Figure 2.42: Example 111.

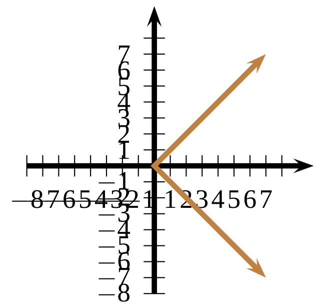

Figure 2.43: Example 112.

111 Example Draw the graph of the curve $y = |1-|x||$.

Solution: ► *The expression* $1-|x|$ *changes sign when* $1-|x| = 0$*, that is, when* $x = \pm 1$*. The expression* $|x|$ *changes sign when* $x = 0$*. Thus we puncture the real line at* $x = -1$*,* $x = 0$ *and* $x = 1$*.*

When $x \leq -1$

$$|1-|x|| = |x|-1 = -x-1.$$

When $-1 \leq x \leq 0$

$$|1-|x|| = 1-|x| = 1+x.$$

When $0 \leq x \leq 1$

$$|1-|x|| = 1-|x| = 1-x.$$

When $x \geq 1$

$$|1-|x|| = |x|-1 = x-1.$$

Hence,

$$y = |1-|x|| = \begin{cases} -x-1 & \text{if } x \leq -1, \\ 1+x & \text{if } -1 \leq x \leq 0, \\ 1-x & \text{if } 0 \leq x \leq 1, \\ x-1 & \text{if } x \geq 1, \end{cases}$$

The graph appears in figure F.6.

◄

112 Example Using Theorem 107, we may deduce that the graph of the curve $x = |y|$ is that which appears in figure F.7

Homework

2.7.1 Problem Consider the curve

$$\mathscr{C} : y = |x-1| - |x| + |x+1|\,.$$

1. Find an expression without absolute values for $\mathscr{C}$ when $x \leq -1$.
2. Find an expression without absolute values for $\mathscr{C}$ when $-1 \leq x \leq 0$.
3. Find an expression without absolute values for $\mathscr{C}$ when $0 \leq x \leq 1$.
4. Find an expression without absolute values for $\mathscr{C}$ when $x \geq 1$.
5. Draw $\mathscr{C}$.

2.7.2 Problem Draw the graph of the curve of equation $|x| = |y|$.

2.7.3 Problem Draw the graph of the curve of equation $y = \dfrac{|x|+x}{2}$.

2.7.4 Problem Draw the plane region

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 16, |x| + |y| \geq 4\}.$$

2.7.5 Problem Draw the graphs of the following equations.

1. $y = |x+2|$
2. $y = 3 - |x+2|$
3. $y = 2|x+2|$
4. $y = |x-1| + |x+1|$
5. $y = |x-1| - |x+1|$
6. $y = |x+1| - |x-1|$
7. $y = |x-1| + |x| + |x+1|$
8. $y = |x-1| - |x| + |x+1|$
9. $y = |x-1| + x + |x+1|$
10. $y = |x+3| + 2|x-1| - |x-4|$

2.8 Parabolas, Hyperbolas, and Ellipses

113 Definition A *parabola* is the collection of all the points on the plane whose distance from a fixed point F (called the *focus* of the parabola) is equal to the distance to a fixed line L (called the *directrix* of the parabola). See figure 2.44, where $FD = DP$.

We can draw a parabola as follows. Cut a piece of thread as long as the trunk of T-square (see figure 2.45). Tie one end to the end of the trunk of the T-square and tie the other end to the focus, say, using a peg. Slide the crosspiece of the T-square along the directrix, while maintaining the thread tight against the ruler with a pencil.

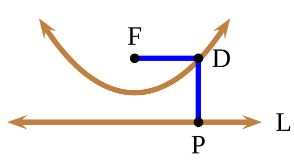

Figure 2.44: Definition of a parabola.

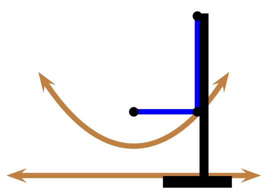

Figure 2.45: Drawing a parabola.

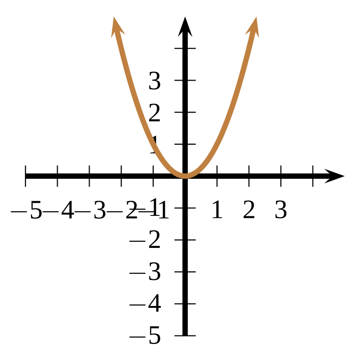

Figure 2.46: Example 115.

114 Theorem Let $d > 0$ be a real number. The equation of a parabola with focus at $(0,d)$ and directrix $y = -d$ is $y = \dfrac{x^2}{4d}$.

Proof: *Let (x,y) be an arbitrary point on the parabola. Then the distance of (x,y) to the line $y=-d$ is $|y+d|$. The distance of (x,y) to the point $(0,d)$ is $\sqrt{x^2+(y-d)^2}$. We have*

$$\begin{aligned} |y+d| = \sqrt{x^2+(y-d)^2} \quad & \implies \quad (|y+d|)^2 = x^2+(y-d)^2 \\ & \implies \quad y^2+2yd+d^2 = x^2+y^2-2yd+d^2 \\ & \implies \quad 4dy = x^2 \\ & \implies \quad y = \frac{x^2}{4d}, \end{aligned}$$

as wanted. ❑

Observe that the midpoint of the perpendicular line segment from the focus to the directrix is on the parabola. We call this point the vertex. *For the parabola $y=\dfrac{x^2}{4d}$ of Theorem 114, the vertex is clearly $(0,0)$.*

115 Example Draw the parabola $y=x^2$.

Solution: ► *From Theorem 114, we want $\dfrac{1}{4d}=1$, that is, $d=\dfrac{1}{4}$. Following Theorem 114, we locate the focus at $(0,\frac{1}{4})$ and the directrix at $y=-\dfrac{1}{4}$ and use a T-square with these references. The vertex of the parabola is at $(0,0)$. The graph is in figure 2.46.* ◄

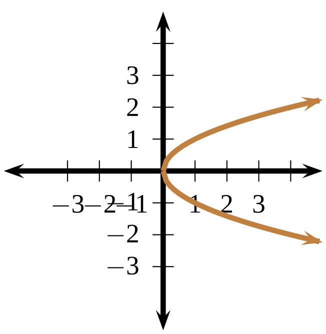

Figure 2.47: $x=y^2$.

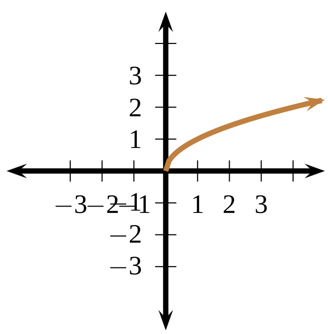

Figure 2.48: $y=\sqrt{x}$.

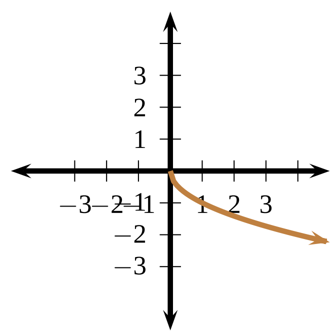

Figure 2.49: $y=-\sqrt{x}$.

116 Example Using Theorem 107, we may draw the graph of the curve $x=y^2$. Its graph appears in figure 2.47.

117 Example Taking square roots on $x=y^2$, we obtain the graphs of $y=\sqrt{x}$ and of $y=-\sqrt{x}$. Their graphs appear in figures 2.48 and 2.49.

118 Definition A *hyperbola* is the collection of all the points on the plane whose absolute value of the difference of the distances from two distinct fixed points F_1 and F_2 (called the *foci*[2] of the hyperbola) is a positive constant. See figure 2.50, where $|F_1D - F_2D| = |F_1D' - F_2D'|$.

We can draw a hyperbola as follows. Put tacks on F_1 and F_2 and measure the distance F_1F_2. Attach piece of thread to one end of the ruler, and the other to F_2, while letting the other end of the ruler to pivot around F_1. The lengths of the ruler and the thread must satisfy

$$\text{length of the ruler} - \text{length of the thread} < F_1F_2.$$

[2] *Foci* is the plural of *focus*.

Hold the pencil against the side of the rule and tighten the thread, as in figure 2.51.

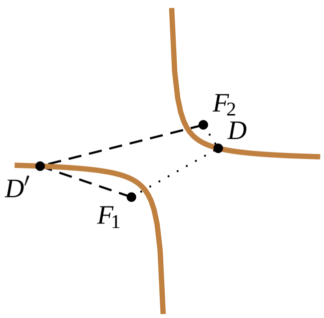

Figure 2.50: Definition of a hyperbola.

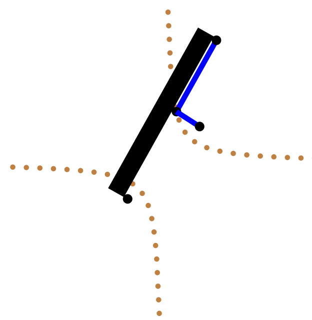

Figure 2.51: Drawing a hyperbola.

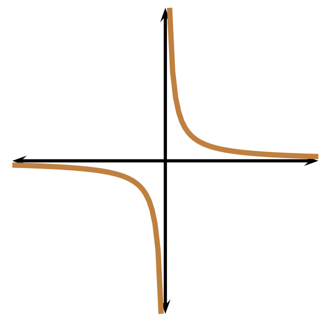

Figure 2.52: The hyperbola $y = \dfrac{1}{x}$.

119 Theorem Let $c > 0$ be a real number. The hyperbola with foci at $F_1 = (-c, -c)$ and $F_2 = (c, c)$, and whose absolute value of the difference of the distances from its points to the foci is $2c$ has equation $xy = \dfrac{c^2}{2}$.

Proof: *Let (x, y) be an arbitrary point on the hyperbola. Then*

$$
\begin{aligned}
& |\mathbf{d}\langle (x,y),(-c,-c)\rangle - \mathbf{d}\langle (x,y),(c,c)\rangle| = 2c \\
\iff\ & \left|\sqrt{(x+c)^2+(y+c)^2} - \sqrt{(x-c)^2+(y-c)^2}\right| = 2c \\
\iff\ & (x+c)^2+(y+c)^2+(x-c)^2+(y-c)^2 - 2\sqrt{(x+c)^2+(y+c)^2}\cdot\sqrt{(x-c)^2+(y-c)^2} = 4c^2 \\
\iff\ & 2x^2+2y^2 = 2\sqrt{(x^2+y^2+2c^2)+(2xc+2yc)}\cdot\sqrt{(x^2+y^2+2c^2)-(2xc+2yc)} \\
\iff\ & 2x^2+2y^2 = 2\sqrt{(x^2+y^2+2c^2)^2-(2xc+2yc)^2} \\
\iff\ & (2x^2+2y^2)^2 = 4\left((x^2+y^2+2c^2)^2-(2xc+2yc)^2\right) \\
\iff\ & 4x^4+8x^2y^2+4y^4 = 4((x^4+y^4+4c^4+2x^2y^2+4y^2c^2+4x^2c^2)-(4x^2c^2+8xyc^2+4y^2c^2)) \\
\iff\ & xy = \frac{c^2}{2},
\end{aligned}
$$

where we have used the identities

$$(A+B+C)^2 = A^2+B^2+C^2+2AB+2AC+2BC \qquad \text{and} \qquad \sqrt{A-B}\cdot\sqrt{A+B} = \sqrt{A^2-B^2}.$$

❑

Observe that the points $\left(-\dfrac{c}{\sqrt{2}}, -\dfrac{c}{\sqrt{2}}\right)$ *and* $\left(\dfrac{c}{\sqrt{2}}, \dfrac{c}{\sqrt{2}}\right)$ *are on the hyperbola* $xy = \dfrac{c^2}{2}$. *We call these points the* vertices[3] *of the hyperbola* $xy = \dfrac{c^2}{2}$.

120 Example To draw the hyperbola $y = \dfrac{1}{x}$ we proceed as follows. According to Theorem 119, its two foci are at $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$. Put length of the ruler − length of the thread $= 2\sqrt{2}$. By alternately pivoting about these points using the procedure above, we get the picture in figure 2.52.

[3] *Vertices* is the plural of *vertex*.

121 Definition An *ellipse* is the collection of points on the plane whose sum of distances from two fixed points, called the *foci*, is constant.

122 Theorem The equation of an ellipse with foci $F_1 = (h-c,k)$ and $F_2 = (h+c,k)$ and sum of distances is the constant $t = 2a$ is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

where $b^2 = a^2 - c^2$.

Proof: *By the triangle inequality, $t > F_1F_2 = 2c$, from where $a > c$. It follows that*

$$\begin{aligned}
& \mathbf{d}\langle(x,y),(x_1,y_1)\rangle + \mathbf{d}\langle(x,y),(x_2,y_2)\rangle = t \\
\iff \quad & \sqrt{(x-h+c)^2+(y-k)^2} = 2a - \sqrt{(x-h-c)^2+(y-k)^2} \\
\iff \quad & (x-h+c)^2+(y-k)^2 = 4a^2 - 4a\sqrt{(x-h-c)^2+(y-k)^2} + (x-h-c)^2+(y-k)^2 \\
\iff \quad & (x-h)^2+2c(x-h)+c^2 = 4a^2 - 4a\sqrt{(x-h-c)^2+(y-k)^2} + (x-h)^2-2c(x-h)+c^2 \\
\iff \quad & (x-h)c - a^2 = -a\sqrt{(x-h-c)^2+(y-k)^2} \\
\iff \quad & (x-h)^2c^2 - 2a^2c(x-h) + a^2 = a^2(x-h-c)^2 + a^2(y-k)^2 \\
\iff \quad & (x-h)^2c^2 - 2a^2c(x-h) + a^2 = a^2(x-h)^2 - 2a^2c(x-h) + a^2c^2 + a^2(y-k)^2 \\
\iff \quad & (x-h)^2(c^2-a^2) - a^2(y-k)^2 = a^2c^2 - a^2 \\
\iff \quad & \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2-c^2} = 1.
\end{aligned}$$

Since $a^2 - c^2 > 0$, we may let $b^2 = a^2 - c^2$, obtaining the result❑

123 Definition The line joining $(h+a,k)$ and $(h-a,k)$ is called the *horizontal axis* of the ellipse and the line joining $(h,k-b)$ and $(h,k+b)$ is called the *vertical axis* of the ellipse. $\max(a,b)$ is the *semi-major axis* and $\min(a,b)$ the *semi-minor axis*.

STOP *The canonical equation of an ellipse whose semi-axes are parallel to the coordinate axes is thus*

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

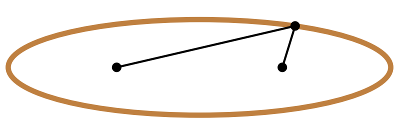

Figure 2.53: Drawing an ellipse.

Figure 2.53 shews how to draw an ellipse by putting tags on the foci, tying the ends of a string to them and tightening the string with a pencil.

124 Example The curve of equation $9x^2 - 18x + 4y^2 + 8y = 23$ is an ellipse, since, by completing squares,

$$9(x^2-2x+1)+4(y^2+2y+1) = 23+9+4 \implies 9(x-1)^2+4(y+1)^2 = 36 \implies \frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1.$$

The centre of the ellipse is $(h,k) = (1,-1)$. The semi-major axis measures $\sqrt{9} = 3$ units and the semi-minor axis measures $\sqrt{4} = 2$ units.

Homework

2.8.1 Problem Let $d > 0$ be a real number. Prove that the equation of a parabola with focus at $(d,0)$ and directrix $x = -d$ is $x = \frac{y^2}{4d}$.

2.8.2 Problem Find the focus and the directrix of the parabola $x = y^2$.

2.8.3 Problem Find the equation of the parabola with directrix $y = -x$ and vertex at $(1,1)$.

2.8.4 Problem Draw the curve $x^2 + 2x + 4y^2 - 8y = 4$.

2.8.5 Problem The point (x,y) moves on the plane in such a way that it is equidistant from the point $(2,3)$ and the line $x = -4$. Find the equation of the curve it describes.

2.8.6 Problem The points $A(0,0)$, B, and C lie on the parabola $y = \frac{x^2}{2}$ as shewn in figure 2.54. If $\triangle ABC$ is equilateral, determine the coordinates of B and C.

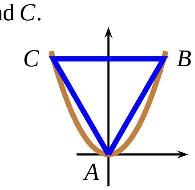

Figure 2.54: Problem 2.8.6.

3 Functions

This chapter introduces the central concept of a *function*. We will only concentrate on functions defined by algebraic formulæ with inputs and outputs belonging to the set of real numbers. We will introduce some basic definitions and will concentrate on the algebraic aspects, as they pertain to formulæ of functions. The subject of *graphing* functions will be taken in subsequent chapters.

3.1 Basic Definitions

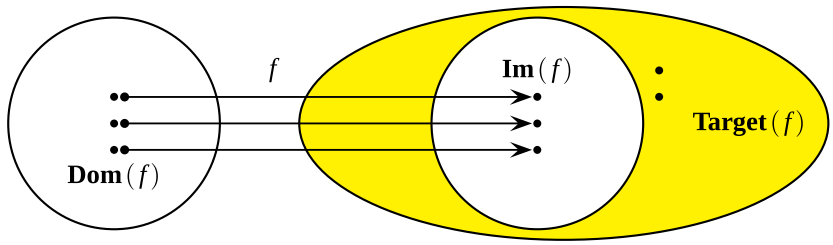

Figure 3.1: The main ingredients of a function.

125 Definition By a *(real-valued) function* $f : \begin{array}{ccc} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x) \end{array}$ we mean the collection of the following ingredients:

1. a *name* for the function. Usually we use the letter f.
2. a set of real number inputs—usually an interval or a finite union of intervals—called the *domain* of the function. The domain of f is denoted by $\mathbf{Dom}(f)$.
3. an *input parameter* , also called *independent variable* or *dummy variable*. We usually denote a typical input by the letter x.
4. a set of possible real number outputs—usually an interval or a finite union of intervals—of the function, called the *target set* of the function. The target set of f is denoted by $\mathbf{Target}(f)$.
5. an *assignment rule* or *formula*, assigning to **every input** a **unique** output. This assignment rule for f is usually denoted by $x \mapsto f(x)$. The output of x under f is also referred to as the *image of x under f*, and is denoted by $f(x)$.

See figure 3.1.

126 Definition Colloquially, we refer to the "function f" when all the other descriptors of the function are understood.

127 Definition The *image* of a function $f : \begin{array}{ccc} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x) \end{array}$ is the set

$$\mathbf{Im}(f) = \{f(x) : x \in \mathbf{Dom}(f)\},$$

that is, the collection of all outputs of f.

Necessarily we have $\mathbf{Im}(f) \subseteq \mathbf{Target}(f)$, *but we will see later on that these two sets may not be equal.*

128 Example Find all functions with domain $\{a,b\}$ and target set $\{c,d\}$.

Solution: ► *Since there are two choices for the output of a and two choices for the output of b, there are* $2^2 = 4$ *such functions, namely:*

1. f_1 *given by* $f_1(a) = f_1(b) = c$. *Observe that* $\mathbf{Im}(f_1) = \{c\}$.
2. f_2 *given by* $f_2(a) = f_2(b) = d$. *Observe that* $\mathbf{Im}(f_2) = \{d\}$.
3. f_3 *given by* $f_3(a) = c, f_3(b) = d$. *Observe that* $\mathbf{Im}(f_1) = \{c,d\}$.
4. f_4 *given by* $f_4(a) = d, f_4(b) = c$. *Observe that* $\mathbf{Im}(f_1) = \{c,d\}$.

◄

It is easy to see that if A has n elements and B has m elements, then the number of functions from A to B is m^n. *For, if* $a_1, a_2, \ldots, a_n$ *are the elements of A, then there are m choices for the output of* a_1, *m choices for the output of* a_2, ..., *m choices for the output of* a_n, *giving a total of*

$$\underbrace{m\cdots m}_{n \text{ times}} = m^n.$$

possibilities.

In some computer programming languages like C, C++, and Java, one defines functions by statements like `int f(double)`. This tells the computer that the input set is allocated enough memory to take a double (real number) variable, and that the output will be allocated enough memory to carry an integer variable.

129 Example Consider the function

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}.$$

Find the following:

1. $f(0)$
2. $f(-\sqrt{2})$
3. $f(1-\sqrt{2})$
4. What is $\mathbf{Im}(f)$?

Solution: ► *We have*

1. $f(0) = 0^2 = 0$
2. $f(-\sqrt{2}) = (-\sqrt{2})^2 = 2$
3. $f(1-\sqrt{2}) = (1-\sqrt{2})^2 = 1^2 - 2\cdot 1\cdot\sqrt{2} + (\sqrt{2})^2 = 3 - 2\sqrt{2}$
4. *Since the square of every real number is positive, we have* $\mathbf{Im}(f) \subseteq [0;+\infty[$. *Now, let* $a \in [0;+\infty[$. *Then* $\sqrt{a} \in \mathbb{R}$ *and* $f(\sqrt{a}) = a$, *so* $a \in \mathbf{Im}(f)$. *This means that* $[0;+\infty[\subseteq \mathbf{Im}(f)$. *We conclude that* $\mathbf{Im}(f) = [0;+\infty[$.

◄

In the above example it was relatively easy to determine the image of the function. In most cases, this calculation is in fact very difficult. This is the reason why in the definition of a function we define the target set to be the set of all *possible outputs*, not the actual outputs. The target set must be large enough to accommodate all the possible outputs of a function.

130 Example Does

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & x^2 \end{array}.$$

define a function?

> **Solution:** ► *No. The target set is not large enough to accommodate all the outputs. The above rule is telling us that every output belongs to* $\mathbb{Z}$*. But this is not true, since for example,* $f(1-\sqrt{2}) = 3-2\sqrt{2} \not\in \mathbb{Z}$*.* ◄

Upon consideration of the preceding example, the reader may wonder why not then, select as target set the entire set $\mathbb{R}$. This is in fact what is done in practice, at least in Calculus. From the point of view of Computer Programming, this is wasteful, as we would be allocating more memory than really needed. When we introduce the concept of *surjections* later on in the chapter, we will see the importance of choosing an appropriate target set.

131 Example Does

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x^2} \end{array}.$$

define a function?

> **Solution:** ► *No. In a function, every input must have a defined output. Since* $f(0)$ *is undefined, this is not a function.* ◄

132 Definition (Equality of Functions) Two functions are equal if

1. Their domains are identical.
2. Their target sets are identical.
3. Their assignment rules are identical.

This means that the only two things that can be different are the names of the functions and the name of the input parameter.

133 Example Consider the functions

$$f : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ x & \mapsto & x^2 \end{array} \qquad g : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ s & \mapsto & s^2 \end{array} \qquad h : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}.$$

Then the functions f and g are the same function. The functions f and h are different functions, as their target sets are different.

We must pay special attention to the fact that although a formula may make sense for a "special input", the "input" may not be part of the domain of the function.

134 Example Consider the function

$$f : \begin{array}{ccc} \mathbb{N}\setminus\{0\} & \to & \mathbb{Q} \\ x & \mapsto & \dfrac{1}{x+\dfrac{1}{x}} \end{array}.$$

Determine:

1. $f(1)$
2. $f(2)$
3. $f\left(\frac{1}{2}\right)$
4. $f(-1)$

Solution: ►

1. $f(1) = \dfrac{1}{1+\dfrac{1}{1}} = \dfrac{1}{2}$

2. $f(2) = \dfrac{1}{2+\dfrac{1}{2}} = \dfrac{1}{\dfrac{5}{2}} = \dfrac{2}{5}$

3. $f\left(\dfrac{1}{2}\right) = \dfrac{1}{\dfrac{1}{2}+\dfrac{1}{\dfrac{1}{2}}} = \dfrac{1}{\dfrac{1}{2}+2} = \dfrac{2}{5}$

4. $f(-1)$ *is undefined, as* $-1 \not\in \mathbb{N}\setminus\{0\}$*, that is* -1 *is not part of the domain.*

◄

It must be emphasised that the exhaustion of the elements of the domain is crucial in the definition of a function. For example, the diagram in figure 3.2 *does not* represent a function, as some elements of the domain are not assigned. Also important in the definition of a function is the fact that the output must be unique. For example, the diagram in 3.3 *does not* represent a function, since the last element of the domain is assigned to two outputs.

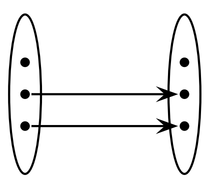

Figure 3.2: Not a function.

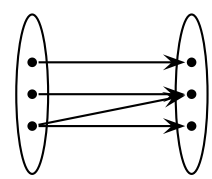

Figure 3.3: Not a function.

To conclude this section, we will give some miscellaneous examples on evaluation of functions.

135 Example (The Identity Function) Consider the function

$$\mathbf{Id} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}.$$

This function assigns to every real its own value. Thus $\mathbf{Id}\,(-1) = -1$, $\mathbf{Id}\,(0) = 0$, $\mathbf{Id}\,(4) = 4$, etc.

STOP *In general, if* $A \subseteq \mathbb{R}$*, the identity function on the set* A *is defined and denoted by*

$$\mathbf{Id}_A : \begin{array}{ccc} A & \to & A \\ x & \mapsto & x \end{array}.$$

136 Example Let $\gamma : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 - 2 \end{array}$. Find $\gamma(x^2+1) - \gamma(x^2-1)$.

Solution: ► *We have*

$$\gamma(x^2+1) - \gamma(x^2-1) = ((x^2+1)^2 - 2) - ((x^2-1)^2 - 2) = (x^4 + 2x^2 + 1 - 2) - (x^4 - 2x^2 + 1 - 2) = 4x^2.$$

◄

Sometimes the assignment rule of a function varies through various subsets of its domain. We call any such function a *piecewise-defined function*.

137 Example Consider the function $f : [-5;4] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 2x \in [-5;1[\\ 2 & \text{if } x = 1 \\ x+1 & \text{if } x \in]1;4] \end{cases}$$

Determine $f(-3)$, $f(1)$, $f(4)$ and $f(5)$.

Solution: ► *Plainly,* $f(-3) = 2(-3) = -6$, $f(1) = 2$, $f(4) = 4 + 1 = 5$, *and* $f(5)$ *is undefined.* ◄

138 Example Write $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |2x - 1|$ as a piecewise-defined function.

Solution: ► *We have* $f(x) = 2x - 1$ *for* $2x - 1 \geq 0$ *and* $f(x) = -(2x - 1)$ *for* $2x - 1 < 0$. *This gives*

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \leq \frac{1}{2} \\ 1 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

◄

Lest the student think that evaluation of functions is a simple affair, let us consider the following example.

139 Example Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(2x+4) = x^2 - 2$. Find

1. $f(6)$
2. $f(1)$
3. $f(x)$
4. $f(f(x))$

Solution: ► *Since* $2x+4$ *is what is inside the parentheses in the formula given, we need to make all inputs equal to it.*

1. *We need* $2x + 4 = 6 \implies x = 1$. *Hence*

$$f(6) = f(2(1)+4) = 1^2 - 2 = -1.$$

2. *We need* $2x + 4 = 1 \implies x = -\frac{3}{2}$. *Hence*

$$f(1) = f\left(2\left(-\frac{3}{2}\right) + 4\right) = \left(-\frac{3}{2}\right)^2 - 2 = \frac{1}{4}.$$

3. *Here we confront a problem. If we proceeded blindly as before and set* $2x+4=x$*, we would get* $x=-4$*, which does not help us much, because what we are trying to obtain is* $f(x)$ for every value of x*. The key observation is that the dummy variable has no idea of what one is calling it, hence, we may first rename the dummy variable: say* $f(2u+4)=u^2-2$*. We need* $2u+4=x \implies u=\dfrac{x-4}{2}$*. Hence*

$$f(x)=f\left(2\left(\frac{x-4}{2}\right)+4\right)=\left(\frac{x-4}{2}\right)^2-2=\frac{x^2}{4}-2x+2.$$

4. *Using the above part,*

$$\begin{aligned} f(f(x)) &= \frac{(f(x))^2}{4}-2f(x)+2 \\ &= \frac{\left(\frac{x^2}{4}-2x+2\right)^2}{4}-2\left(\frac{x^2}{4}-2x+2\right)+2 \\ &= \frac{x^4}{64}-\frac{x^3}{4}+\frac{3x^2}{4}+2x-1 \end{aligned}$$

◀

140 Example $f:\mathbb{R}\to\mathbb{R}$ is a function satisfying $f(3)=2$ and $f(x+3)=f(3)f(x)$. Find $f(-3)$.

Solution: ▶ *Since we are interested in* $f(-3)$*, we first put* $x=-3$ *in the relation, obtaining*

$$f(0)=f(3)f(-3).$$

Thus we must also know $f(0)$ *in order to find* $f(-3)$*. Letting* $x=0$ *in the relation,*

$$f(3)=f(3)f(0) \implies f(3)=f(3)f(3)f(-3) \implies 2=4f(-3) \implies f(-3)=\frac{1}{2}.$$

◀

The following example is a surprising application of the concept of function.

141 Example Consider the polynomial $(x^2-2x+2)^{2008}$. Find its constant term. Also, find the sum of its coefficients after the polynomial has been expanded and like terms collected.

Solution: ▶ *The polynomial has degree* $2\cdot 2008=4016$*. This means that after expanding out, it can be written in the form*

$$(x^2-2x+2)^{2008}=a_0x^{4016}+a_1x^{4015}+\cdots+a_{4015}x+a_{4016}.$$

Consider now the function

$$p: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & a_0x^{4016}+a_1x^{4015}+\cdots+a_{4015}x+a_{4016} \end{array}.$$

The constant term of the polynomial is a_{4016}*, which happens to be* $p(0)$*. Hence the constant term is*

$$a_{4016}=p(0)=(0^2-2\cdot 0+2)^{2008}=2^{2008}.$$

The sum of the coefficients of the polynomial is

$$a_0+a_1+a_2+\cdots+a_{4016}=p(1)=(1^2-2\cdot 1+2)^{2008}=1.$$

◀

Homework

3.1.1 Problem Let

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{x-1}{x^2+1} \end{array}.$$

Find $f(0)+f(1)+f(2)$ and $f(0+1+2)$. Is it true that

$$f(0)+f(1)+f(2) = f(0+1+2)\ ?$$

Is there a real solution to the equation $f(x) = \frac{1}{x}$? Is there a real solution to the equation $f(x) = x$?

3.1.2 Problem Find all functions from $\{0,1,2\}$ to $\{-1,1\}$.

3.1.3 Problem Find all functions from $\{-1,1\}$ to $\{0,1,2\}$.

3.1.4 Problem Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2 - x$. Find

$$\frac{f(x+h)-f(x-h)}{h}.$$

3.1.5 Problem Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3 - 3x$. Find

$$\frac{f(x+h)-f(x-h)}{h}.$$

3.1.6 Problem Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Which of the following statements are always true?

1. $f\left(\frac{a}{b}\right) = \frac{f(a)}{f(b)}$.
2. $f(a+b) = f(a)+f(b)$.
3. $f(a^2) = (f(a))^2$

3.1.7 Problem Let $a : \mathbb{R} \to \mathbb{R}$, be given by $a(2-x) = x^2 - 5x$. Find $a(3)$, $a(x)$ and $a(a(x))$.

3.1.8 Problem Let $f : \mathbb{R} \to \mathbb{R}$, $f(1-x) = x^2 - 2$. Find $f(-2)$, $f(x)$ and $f(f(x))$.

3.1.9 Problem Let $f : \mathbf{Dom}(f) \to \mathbb{R}$ be a function. f is said to have a *fixed point* at $t \in \mathbf{Dom}(f)$ if $f(t) = t$. Let $s : [0;+\infty[\to \mathbb{R}$, $s(x) = x^5 - 2x^3 + 2x$. Find all fixed points of s.

3.1.10 Problem Let $: \mathbb{R} \to \mathbb{R}$, $h(x+2) = 1 + x - x^2$. Express $h(x-1)$, $h(x)$, $h(x+1)$ as powers of x.

3.1.11 Problem Let $f : \mathbb{R} \to \mathbb{R}$, $f(x+1) = x^2$. Find $f(x)$, $f(x+2)$ and $f(x-2)$ as powers of x.

3.1.12 Problem Let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(1-x) = 2x$. Find $h(3x)$.

3.1.13 Problem Consider the polynomial

$$(1-x^2+x^4)^{2003} = a_0 + a_1x + a_2x^2 + \cdots + a_{8012}x^{8012}.$$

Find

1. a_0
2. $a_0 + a_1 + a_2 + \cdots + a_{8012}$
3. $a_0 - a_1 + a_2 - a_3 + \cdots - a_{8011} + a_{8012}$
4. $a_0 + a_2 + a_4 + \cdots + a_{8010} + a_{8012}$
5. $a_1 + a_3 + \cdots + a_{8009} + a_{8011}$

3.1.14 Problem Let $f : \mathbb{R} \to \mathbb{R}$, be a function such that $\forall x \in]0;+\infty[$,

$$[f(x^3+1)]^{\sqrt{x}} = 5,$$

find the value of

$$\left[f\left(\frac{27+y^3}{y^3}\right)\right]^{\sqrt{\frac{27}{y}}}$$

for $y \in]0;+\infty[$.

3.2 Graphs of Functions and Functions from Graphs

In this section we *briefly* describe graphs of functions. The bulk of graphing will be taken up in subsequent chapters, as graphing functions with a given formula is a very tricky matter.

142 Definition The *graph* of a function $f : \begin{array}{ccc} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x) \end{array}$ is the set $\Gamma_f = \{(x,y) \in \mathbb{R}^2 : y = f(x)\}$ on the plane.

For ellipsis, we usually say *the graph of f*, or *the graph $y = f(x)$* or the *the curve $y = f(x)$*.

By the definition of the graph of a function, the x-axis contains the set of inputs and y-axis has the set of outputs. Since in the definition of a function every input goes to exactly one output, wee see that *if a vertical line crosses two or more points of a graph, the graph does not represent a function*. We will call this the *vertical line test* for a function. See figures 3.4 and 3.5.

At this stage there are very few functions with a given formula and infinite domain that we know how to graph. Let us list some of them.

143 Example (Identity Function) Consider the function

$$\mathbf{Id}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}.$$

By Theorem 93, the graph of the identity function is a straight line.

144 Example (Absolute Value Function) Consider the function

$$\mathbf{AbsVal}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & |x| \end{array}.$$

By Example 108, the graph of the absolute value function is that which appears in figure 3.7.

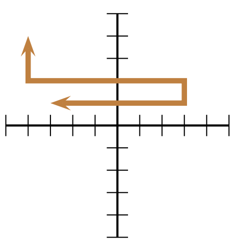

Figure 3.4: Fails the vertical line test. Not a function.

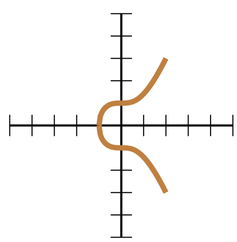

Figure 3.5: Fails the vertical line test. Not a function.

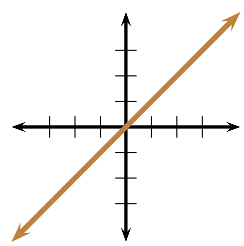

Figure 3.6: **Id**

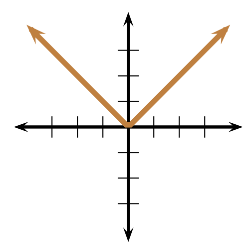

Figure 3.7: **AbsVal**

145 Example (The Square Function) Consider the function

$$\mathbf{Sq}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}.$$

This function assigns to every real its square. By Theorem 114, the graph of the square function is a parabola, and it is presented in in figure 3.8.

146 Example (The Square Root Function) Consider the function

$$\mathbf{Rt}: \begin{array}{ccc} [0;+\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x} \end{array}.$$

By Example 117, the graph of the square root function is the half parabola that appears in figure 3.9.

147 Example (Semicircle Function) Consider the function[1]

$$\mathbf{Sc}: \begin{array}{ccc} [-1;1] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{1-x^2} \end{array}.$$

By Example 89, the graph of **Sc** is the upper unit semicircle, which is shewn in figure 3.10.

[1] Since we are concentrating exclusively on real-valued functions, the formula for **Sc** only makes sense in the interval $[-1;1]$. We will examine this more closely in the next section.

148 Example (The Reciprocal function) Consider the function[2]

$$\mathbf{Rec}: \begin{array}{ccc} \mathbb{R}\setminus\{0\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x} \end{array}.$$

By Example 120, the graph of the reciprocal function is the hyperbola shewn in figure 3.11.

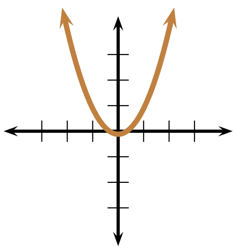

Figure 3.8: **Sq**

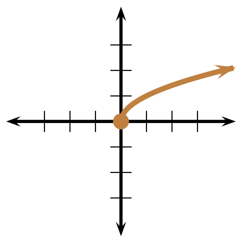

Figure 3.9: **Rt**

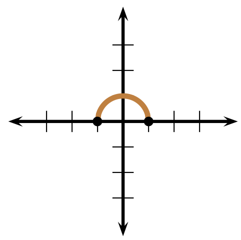

Figure 3.10: **Sc**

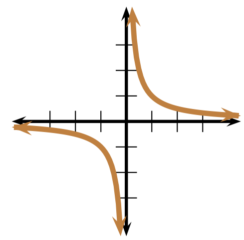

Figure 3.11: **Rec**

We can combine pieces of the above curves in order to graph piecewise defined functions.

149 Example Consider the function $f : \mathbb{R}\setminus\{-1,1\} \to \mathbb{R}$ with assignment rule

$$f(x) = \begin{cases} -x & \text{if } x < -1 \\ x^2 & \text{if } -1 < x < 1 \\ x & \text{if } x > 1 \end{cases}$$

Its graph appears in figure 3.12.

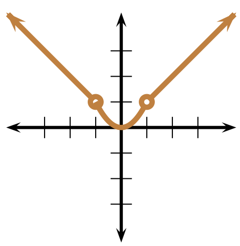

Figure 3.12: Example 149.

The alert reader will notice that, for example, the two different functions

$$f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array} \qquad g: \begin{array}{ccc} \mathbb{R} & \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}$$

possess the same graph. It is then difficult to recover all the information about a function from its graph, in particular, it is impossible to recover its target set. We will now present a related concept in order to alleviate this problem.

150 Definition A *functional curve* on the plane is a curve that passes the vertical line test. The *domain of the functional curve* is the "shadow" of the graph on the x-axis, and the *image of the functional curve* is its shadow on the y-axis.

[2]The formula for **Rec** only makes sense when $x \neq 0$.

In order to distinguish between finite and infinite sets, we will make the convention that arrow heads in a functional curve indicate that the curve continues to infinity in te direction of the arrow. In order to indicate that a certain value is not part of the domain, we will use a hollow dot. Also, in order to make our graphs readable, we will assume that endpoints and dots fall in lattice points, that is, points with integer coordinates. The following example will elaborate on our conventions.

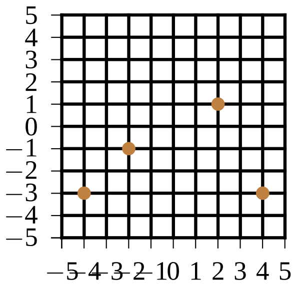

Figure 3.13: Example 151: *a*.

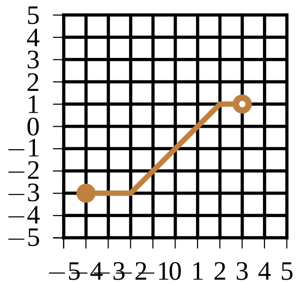

Figure 3.14: Example 151: *b*.

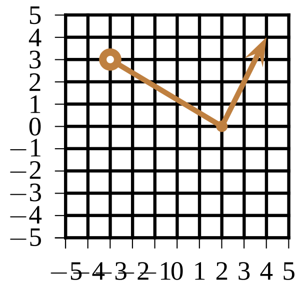

Figure 3.15: Example 151: *c*.

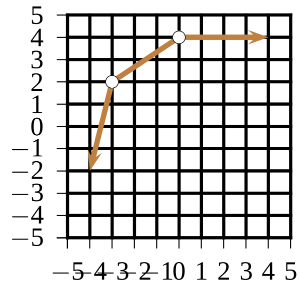

Figure 3.16: Example 151: *d*.

151 Example Determine the domains and images of the functional curves a,b,c,d given in figures 3.13 through 3.16.

> **Solution:** ► *Figure 3.13 consists only a finite number of dots. These dots x-coordinates are the set* $\{-4,-2,2,4\}$ *and hence* $\mathbf{Dom}(a) = \{-4,-2,2,4\}$*. The dots y-coordinates are the set* $\{-3,-1,1\}$ *and so* $\mathbf{Im}(a) = \{-3.-1,1\}$.
>
> *Figure 3.14 has x-shadow on the interval* $[-3;3[$*. Notice that* $x = 3$ *is excluded since it has an open dot. We conclude that* $\mathbf{Dom}(b) = [-3;3[$*. The y-shadow of this set is the interval* $[-3;1]$*. Notice that we do include* $y = 1$ *since there are points having y-coordinate* 1*, for example* $(2,1)$*, which are on the graph. Hence,* $\mathbf{Im}(b) = [-3;1]$.
>
> *The x-shadow of figure 3.15 commences just right of* $x = -3$ *and extends to* $+\infty$*, as we have put an arrow on the rightmost extreme of the curve. Hence* $\mathbf{Dom}(c) =]-3:+\infty[$*. The y-shadow of this curve starts at* $y = 0$ *and continues to* $+\infty$*, thus* $\mathbf{Im}(c) = [0;+\infty[$.
>
> *We leave to the reader to conclude from figure 3.16 that*
>
> $$\mathbf{Dom}(d) = \mathbb{R} \setminus \{-3,0\} =]-\infty;-3[\cup]-3;0[\cup]0;+\infty[, \qquad \mathbf{Im}(d) =]-\infty;2[\cup]2;4].$$
>
> ◄

Homework

3.2.1 Problem Consider the functional curve d shewn in figure 3.16.

1. Find consecutive integers a,b such that $d(-2) \in [a;b]$.
2. Determine $d(-3)$.
3. Determine $d(0)$.
4. Determine $d(100)$.

3.2.2 Problem The *signum* function is defined as follows:

$$\textbf{signum}: \begin{array}{ccc} \mathbb{R} & \to & \{-1,0,1\} \\ x & \mapsto & \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \end{array}.$$

Graph the signum function.

3.2.3 Problem By looking at the graph of the identity function **Id**, determine $\mathbf{Dom}(\mathbf{Id})$ and $\mathbf{Im}(\mathbf{Id})$.

3.2.4 Problem By looking at the graph of the absolute value function **AbsVal**, determine **Dom** (**AbsVal**) and **Im** (**AbsVal**).

3.2.5 Problem By looking at the graph of the square function **Sq**, determine **Dom** (**Sq**) and **Im** (**Sq**).

3.2.6 Problem By looking at the graph of the square root function **Rt**, determine **Dom** (**Rt**) and **Im** (**Rt**).

3.2.7 Problem By looking at the graph of the semicircle function **Sc**, determine **Dom** (**Sc**) and **Im** (**Sc**).

3.2.8 Problem By looking at the graph of the reciprocal function **Rec**, determine **Dom** (**Rec**) and **Im** (**Rec**).

3.2.9 Problem Graph the function $g : \mathbb{R} \to \mathbb{R}$ that is piecewise defined by

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in]-\infty; -1[\\ x & \text{if } x \in [-1; 1] \\ \dfrac{1}{x} & \text{if } x \in]1; +\infty[\end{cases}$$

3.2.10 Problem Consider the function $f : [-4; 4] \to [-5; 1]$ whose graph is made of straight lines, as in figure 3.17. Find a piecewise formula for f.

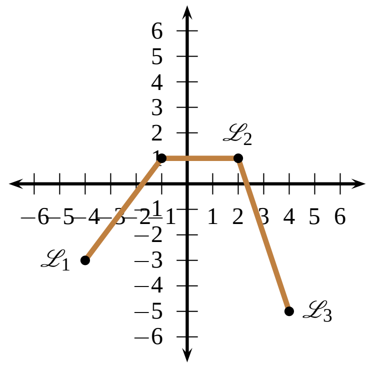

Figure 3.17: Problem 3.2.10.

3.3 Natural Domain of an Assignment Rule

Given a formula, we are now interested in determining which possible subsets of $\mathbb{R}$ will render the output of the formula also a real number subset.

152 Definition The *natural domain of an assignment rule* is the largest set of real number inputs that will give a real number output of a given assignment rule.

For the algebraic combinations that we are dealing with, we must then worry about having non-vanishing denominators and taking even-indexed roots of positive real numbers.

153 Example Find the natural domain of the rule $x \mapsto \dfrac{1}{x^2 - x - 6}$.

Solution: ▶ *In order for the output to be a real number, the denominator must not vanish. We must have* $x^2 - x - 6 = (x+2)(x-3) \neq 0$, *and so* $x \neq -2$ *nor* $x \neq 3$. *Thus the natural domain of this rule is* $\mathbb{R} \setminus \{-2, 3\}$.

◀

154 Example Find the natural domain of $x \mapsto \dfrac{1}{x^4 - 16}$.

Solution: Since $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x+2)(x-2)(x^2+4)$, the rule is undefined when $x = -2$ or $x = 2$. The natural domain is thus $\mathbb{R} \setminus \{-2, +2\}$.

155 Example Find the natural domain for the rule $f(x) = \dfrac{2}{4 - |x|}$.

Solution: ▶ *The denominator must not vanish, hence* $x \neq \pm 4$. *The natural domain of this rule is thus* $\mathbb{R} \setminus \{-4, 4\}$.

◀

156 Example Find the natural domain of the rule $f(x) = \sqrt{x+3}$

Solution: ► *In order for the output to be a real number, the quantity under the square root must be positive, hence* $x+3 \geq 0 \implies x \geq -3$ *and the natural domain is the interval* $[-3;+\infty[$.

◄

157 Example Find the natural domain of the rule $g(x) = \dfrac{2}{\sqrt{x+3}}$

Solution: ► *The denominator must not vanish, and hence the quantity under the square root must be positive, therefore* $x > -3$ *and the natural domain is the interval* $]-3+;\infty[$.

◄

158 Example Find the natural domain of the rule $x \mapsto \sqrt[4]{x^2}$.

Solution: ► *Since for all real numbers* $x^2 \geq 0$, *the natural domain of this rule is* $\mathbb{R}$.

◄

159 Example Find the natural domain of the rule $x \mapsto \sqrt[4]{-x^2}$.

Solution: ► *Since for all real numbers* $-x^2 \leq 0$, *the quantity under the square root is a real number only when* $x = 0$, *whence the natural domain of this rule is* $\{0\}$.

◄

160 Example Find the natural domain of the rule $x \mapsto \dfrac{1}{\sqrt{x^2}}$.

Solution: ► *The denominator vanishes when* $x = 0$. *Otherwise for all real numbers,* $x \neq 0$, *we have* $x^2 > 0$. *The natural domain of this rule is thus* $\mathbb{R} \setminus \{0\}$.

◄

161 Example Find the natural domain of the rule $x \mapsto \dfrac{1}{\sqrt{-x^2}}$.

Solution: ► *The denominator vanishes when* $x = 0$. *Otherwise for all real numbers,* $x \neq 0$, *we have* $-x^2 < 0$. *Thus* $\sqrt{-x^2}$ *is only a real number when* $x = 0$, *and in that case, the denominator vanishes. The natural domain of this rule is thus the empty set* $\varnothing$.

◄

162 Example Find the natural domain of the assignment rule

$$x \mapsto \sqrt{1-x} + \frac{1}{\sqrt{1+x}}.$$

Solution: ► *We need simultaneously* $1-x \geq 0$ *(which implies that* $x \leq 1$*) and* $1+x > 0$ *(which implies that* $x > -1$*), so* $x \in]-1;1]$.

◄

163 Example Find the largest subset of real numbers where the assignment rule $x \mapsto \sqrt{x^2-x-6}$ gives real number outputs.

Solution: ► *The quantity* $x^2-x-6 = (x+2)(x-3)$ *under the square root must be positive. Studying the sign diagram*

$x \in$	$]-\infty;-2]$	$[-2;3]$	$[3;+\infty[$
$\mathrm{signum}\,(x+2) =$	$-$	$+$	$+$
$\mathrm{signum}\,(x-3) =$	$-$	$-$	$+$
$\mathrm{signum}\,((x+2)(x-2)) =$	$+$	$-$	$+$

we conclude that the natural domain of this formula is the set $]-\infty;-2]\cup[3;+\infty[$.

◀

164 Example Find the natural domain for the rule $f(x) = \dfrac{1}{\sqrt{x^2-x-6}}$.

Solution: ▶ *The denominator must not vanish, so the quantity under the square root must be positive. By the preceding problem this happens when* $x \in]-\infty;-2[\cup]3;+\infty[$. ◀

165 Example Find the natural domain of the rule $x \mapsto \sqrt{x^2+1}$.

Solution: ▶ *Since* $\forall x \in \mathbb{R}$ *we have* $x^2+1 \geq 1$, *the square root is a real number for all real* x. *Hence the natural domain is* $\mathbb{R}$. ◀

166 Example Find the natural domain of the rule $x \mapsto \sqrt{x^2+x+1}$.

Solution: ▶ *The discriminant of* $x^2+x+1=0$ *is* $1^2-4(1)(1)<0$. *Since the coefficient of* x^2 *is* $1>0$, *the expression* x^2+x+1 *is always positive, meaning that the required natural domain is all of* $\mathbb{R}$.

Aliter: *Observe that since*

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0,$$

the square root is a real number for all real x. *Hence the natural domain is* $\mathbb{R}$.

◀

Homework

3.3.1 Problem Below are given some assignment rules. Verify that the accompanying set is the natural domain of the assignment rule.

Assignment Rule	Natural Domain
$x \mapsto \sqrt{(1-x)(x+3)}$	$x \in [-3;1]$.
$x \mapsto \sqrt{\dfrac{1-x}{x+3}}$	$x \in]-3;1]$
$x \mapsto \sqrt{\dfrac{x+3}{1-x}}$	$x \in [-3;1[$
$x \mapsto \sqrt{\dfrac{1}{(x+3)(1-x)}}$	$x \in]-3;1[$

3.3.2 Problem Find the natural domain for the given assignment rules.

1. $x \mapsto \dfrac{1}{\sqrt{1+|x|}}$
2. $x \mapsto \sqrt[4]{5-|x|}$
3. $x \mapsto \sqrt[3]{5-|x|}$
4. $x \mapsto \dfrac{1}{x^2+2x+2}$
5. $x \mapsto \dfrac{1}{\sqrt{x^2-2x-2}}$
6. $x \mapsto \dfrac{1}{|x-1|+|x+1|}$
7. $x \mapsto \dfrac{\sqrt{-x}}{x^2-1}$
8. $x \mapsto \dfrac{\sqrt{1-x^2}}{1-|x|}$
9. $x \mapsto \sqrt{x}+\sqrt{-x}$

3.3.3 Problem Below are given some assignment rules. Verify that the accompanying set is the natural domain of the assignment rule.

Assignment Rule	Natural Domain
$x \mapsto \sqrt{\dfrac{x}{x^2-9}}$	$x \in]-3;0] \bigcup]3;+\infty[$
$x \mapsto \sqrt{-\lvert x\rvert}$	$x = 0$
$x \mapsto \sqrt{-\lvert\lvert x\rvert-2\rvert}$	$x \in \{-2,2\}$
$x \mapsto \sqrt{\dfrac{1}{x}}$	$x \in]0;+\infty[$
$x \mapsto \sqrt{\dfrac{1}{x^2}}$	$x \in \mathbb{R} \setminus \{0\}$
$x \mapsto \sqrt{\dfrac{1}{-x}}$	$x \in]-\infty;0[$
$x \mapsto \sqrt{\dfrac{1}{-\lvert x\rvert}}$	$\varnothing$ (the empty set)
$x \mapsto \dfrac{1}{x\sqrt{x+1}}$	$x \in]-1;0[\bigcup]0;+\infty[$
$x \mapsto \sqrt{1+x}+\sqrt{1-x}$	$[-1;1]$

3.3.4 Problem Find the natural domain for the rule $f(x) = \sqrt{x^3-12x}$.

3.3.5 Problem Find the natural domain of the rule $x \mapsto \dfrac{1}{\sqrt{x^2-2x-2}}$.

3.3.6 Problem Find the natural domain for the following rules.

1. $x \mapsto \sqrt{-(x+1)^2}$,
2. $x \mapsto \dfrac{1}{\sqrt{-(x+1)^2}}$
3. $f(x) = \dfrac{x^{1/2}}{\sqrt{x^4-13x^2+36}}$
4. $g(x) = \dfrac{\sqrt[4]{3-x}}{\sqrt{x^4-13x^2+36}}$
5. $h(x) = \dfrac{1}{\sqrt{x^6-13x^4+36x^2}}$
6. $j(x) = \dfrac{1}{\sqrt{x^5-13x^3+36x}}$
7. $k(x) = \dfrac{1}{\sqrt{\lvert x^4-13x^2+36\rvert}}$

3.4 Algebra of Functions

167 Definition Let $f : \mathbf{Dom}(f) \to \mathbf{Target}(f)$ and $g : \mathbf{Dom}(g) \to \mathbf{Target}(g)$. Then $\mathbf{Dom}(f \pm g) = \mathbf{Dom}(f) \cap \mathbf{Dom}(g)$ and the sum (respectively, difference) function $f+g$ (respectively, $f-g$) is given by

$$f \pm g : \begin{array}{ccc} \mathbf{Dom}(f)\cap\mathbf{Dom}(g) & \to & \mathbf{Target}(f\pm g) \\ x & \mapsto & f(x)\pm g(x) \end{array}.$$

In other words, if x belongs both to the domain of f and g, then

$$(f \pm g)(x) = f(x) \pm g(x).$$

168 Definition Let $f : \mathbf{Dom}(f) \to \mathbb{R}$ and $g : \mathbf{Dom}(g) \to \mathbb{R}$. Then $\mathbf{Dom}(fg) = \mathbf{Dom}(f) \cap \mathbf{Dom}(g)$ and the product function fg is given by

$$fg : \begin{array}{ccc} \mathbf{Dom}(f)\cap\mathbf{Dom}(g) & \to & \mathbf{Target}(fg) \\ x & \mapsto & f(x)\cdot g(x) \end{array}.$$

In other words, if x belongs both to the domain of f and g, then

$$(fg)(x) = f(x)\cdot g(x).$$

169 Example Let

$$f : \begin{array}{ccc} [-1;1] & \to & \mathbb{R} \\ x & \mapsto & x^2+2x \end{array}, \quad g : \begin{array}{ccc} [0;2] & \to & \mathbb{R} \\ x & \mapsto & 3x+2 \end{array}.$$

Find

1. $\mathbf{Dom}(f \pm g)$
2. $\mathbf{Dom}(fg)$
3. $(f+g)(-1)$
4. $(f+g)(1)$
5. $(fg)(1)$
6. $(f-g)(0)$

7. $(f+g)(2)$

Solution: ► *We have*

1. $\mathbf{Dom}(f \pm g) = \mathbf{Dom}(f) \cap \mathbf{Dom}(g) = [-1;1] \cap [0;2] = [0;1]$.
2. $\mathbf{Dom}(fg)$ *is also* $\mathbf{Dom}(f) \cap \mathbf{Dom}(g) = [0;1]$.
3. *Since* $-1 \not\in [0;1]$, $(f+g)(-1)$ *is undefined.*
4. $(f+g)(1) = f(1)+g(1) = 3+5 = 8$.
5. $(fg)(1) = f(1)g(1) = (3)(5) = 15$.
6. $(f-g)(0) = f(0)-g(0) = 0-2 = -2$.
7. *Since* $2 \not\in [0;1]$, $(f+g)(2)$ *is undefined.*

◄

170 Definition Let $g : \mathbf{Dom}(g) \to \mathbb{R}$ be a function. The *support* of g, denoted by $\mathbf{supp}(g)$ is the set of elements in $\mathbf{Dom}(g)$ where g does not vanish, that is

$$\mathbf{supp}(g) = \{x \in \mathbf{Dom}(g) : g(x) \neq 0\}.$$

171 Example Let

$$g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3-2x \end{array}.$$

Then $x^3-2x = x(x-\sqrt{2})(x+\sqrt{2})$. Thus

$$\mathbf{supp}(g) = \mathbb{R} \setminus \{-\sqrt{2}, 0\sqrt{2}\}.$$

172 Example Let

$$g : \begin{array}{ccc} [0;1] & \to & \mathbb{R} \\ x & \mapsto & x^3-2x \end{array}.$$

Then $x^3-2x = x(x-\sqrt{2})(x+\sqrt{2})$. Thus

$$\mathbf{supp}(g) = [0;1] \setminus \{-\sqrt{2}, 0\sqrt{2}\} =]0;1].$$

173 Definition Let $f : \mathbf{Dom}(f) \to \mathbb{R}$ and $g : \mathbf{Dom}(g) \to \mathbb{R}$. Then $\mathbf{Dom}\left(\frac{f}{g}\right) = \mathbf{Dom}(f) \cap \mathbf{supp}(g)$ and the quotient function $\frac{f}{g}$ is given by

$$\frac{f}{g} : \begin{array}{ccc} \mathbf{Dom}(f) \cap \mathbf{supp}(g) & \to & \mathbf{Target}\left(\frac{f}{g}\right) \\ x & \mapsto & \frac{f(x)}{g(x)} \end{array}.$$

In other words, if x belongs both to the domain of f and g and $g(x) \neq 0$, then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

174 Example Let

$$f : \begin{array}{ccc} [-2;3] & \to & \mathbb{R} \\ x & \mapsto & x^3-x \end{array}, \quad g : \begin{array}{ccc} [0;5] & \to & \mathbb{R} \\ x & \mapsto & x^3-2x^2 \end{array}.$$

Find

1. $\mathbf{supp}\,(f)$
2. $\mathbf{supp}\,(g)$
3. $\mathbf{Dom}\left(\frac{f}{g}\right)$
4. $\mathbf{Dom}\left(\frac{g}{f}\right)$
5. $\left(\frac{f}{g}\right)(2)$
6. $\left(\frac{g}{f}\right)(2)$
7. $\left(\frac{f}{g}\right)(1/3)$
8. $\left(\frac{g}{f}\right)(1/3)$

Solution: ►

1. *As* $x^3 - x = x(x-1)(x+1)$, $\mathbf{supp}\,(f) = [-2;-1[\cup]-1;0[\cup]0;3]$
2. *As* $x^3 - 2x^2 = x^2(x-2)$, $\mathbf{supp}\,(g) =]0;2[\cup]2;5]$.
3. $\mathbf{Dom}\left(\frac{f}{g}\right) = \mathbf{Dom}\,(f) \cap \mathbf{supp}\,(g) = [-2;3] \cap (]0;2[\cup]2;5]) =]0;2[\cup]2;3]$
4. $$\mathbf{Dom}\left(\frac{g}{f}\right) = \mathbf{Dom}\,(g) \cap \mathbf{supp}\,(f) = [0;5] \cap ([-2;-1[\cup]-1;0[\cup]0;3]) =]0;3]$$
5. $\left(\frac{f}{g}\right)(2)$ *is undefined, as* $2 \notin]0;2[\cup]2;3]$.
6. $\left(\frac{g}{f}\right)(2) = \frac{g(2)}{f(2)} = \frac{0}{6} = 0.$
7. $\left(\frac{f}{g}\right)(1/3) = \frac{-\frac{8}{27}}{-\frac{5}{27}} = \frac{8}{5}$
8. $\left(\frac{g}{f}\right)(1/3) = \frac{-\frac{5}{27}}{-\frac{8}{27}} = \frac{5}{8}$

◄

We are now going to consider "functions of functions."

175 Definition Let $f : \mathbf{Dom}\,(f) \to \mathbf{Target}\,(f)$, $g : \mathbf{Dom}\,(g) \to \mathbf{Target}\,(g)$ and let $U = \{x \in \mathbf{Dom}\,(g) : g(x) \in \mathbf{Dom}\,(f)\}$. We define the *composition* function of f and g as

$$f \circ g : \begin{array}{ccc} U & \to & \mathbf{Target}\,(f \circ g) \\ x & \mapsto & f(g(x)) \end{array}. \tag{3.1}$$

We read $f \circ g$ as "f *composed with* g."

We have $\mathbf{Dom}\,(f \circ g) = \{x \in \mathbf{Dom}\,(g) : g(x) \in \mathbf{Dom}\,(f)\}$. *Thus to find* $\mathbf{Dom}\,(f \circ g)$ *we find those elements of* $\mathbf{Dom}\,(g)$ *whose images are in* $\mathbf{Dom}\,(f) \cap \mathbf{Im}\,(g)$

176 Example Let

$$f : \begin{array}{ccc} \{-2,-1,0,1,2\} & \to & \mathbb{R} \\ x & \mapsto & 2x+1 \end{array}, \quad g : \begin{array}{ccc} \{0,1,2,3\} & \to & \mathbb{R} \\ x & \mapsto & x^2-4 \end{array}.$$

1. Find $\mathbf{Im}\,(f)$.
2. Find $\mathbf{Im}\,(g)$.
3. Find $\mathbf{Dom}\,(f\circ g)$.
4. Find $\mathbf{Dom}\,(g\circ f)$.
5. Find $(f\circ g)(0)$.
6. Find $(g\circ f)(0)$.
7. Find $(f\circ g)(2)$.
8. Find $(g\circ f)(2)$.

Solution: ▶

1. *We have* $f(-2)=-3$, $f(-1)=-1$, $f(0)=1$, $f(1)=3$, $f(2)=5$. *Hence* $\mathbf{Im}\,(f)=\{-3,-1,1,3,5\}$.
2. *We have* $g(0)=-4$, $g(1)=-3$, $g(2)=0$, $g(3)=5$. *Hence* $\mathbf{Im}\,(g)=\{-4,-3,0,5\}$.
3. $\mathbf{Dom}\,(f\circ g)=\{x\in\mathbf{Dom}\,(g) : g(x)\in\mathbf{Dom}\,(f)\}=\{2\}$.
4. $\mathbf{Dom}\,(g\circ f)=\{x\in\mathbf{Dom}\,(f) : f(x)\in\mathbf{Dom}\,(g)\}=\{0,1\}$.
5. $(f\circ g)(0)=f(g(0))=f(-4)$, *but this last is undefined.*
6. $(g\circ f)(0)=g(f(0))=g(1)=-3$.
7. $(f\circ g)(2)=f(g(2))=f(0)=1$.
8. $(g\circ f)(2)=g(f(2))=g(5)$, *but this last is undefined.*

◀

177 Example Let

$$f:\begin{array}{ccc}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & 2x-3\end{array},\quad g:\begin{array}{ccc}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & 5x+1\end{array}.$$

1. Demonstrate that $\mathbf{Im}\,(f)=\mathbb{R}$.
2. Demonstrate that $\mathbf{Im}\,(g)=\mathbb{R}$.
3. Find $(f\circ g)(x)$.
4. Find $(g\circ f)(x)$.
5. Is it ever true that $(f\circ g)(x)=(g\circ f)(x)$?

Solution: ▶

1. *Take* $b\in\mathbb{R}$. *We must shew that* $\exists x\in\mathbb{R}$ *such that* $f(x)=b$. *But*
$$f(x)=b\implies 2x-3=b\implies x=\frac{b+3}{2}.$$
Since $\frac{b+3}{2}$ *is a real number satisfying* $f\left(\frac{b+3}{2}\right)=b$, *we have shewn that* $\mathbf{Im}\,(f)=\mathbb{R}$.
2. *Take* $b\in\mathbb{R}$. *We must shew that* $\exists x\in\mathbb{R}$ *such that* $g(x)=b$. *But*
$$g(x)=b\implies 5x+1=b\implies x=\frac{b-1}{5}.$$
Since $\frac{b-1}{5}$ *is a real number satisfying* $g\left(\frac{b-1}{5}\right)=b$, *we have shewn that* $\mathbf{Im}\,(g)=\mathbb{R}$.
3. *We have*
$$(f\circ g)(x)=f(g(x))=f(5x+1)=2(5x+1)-3=10x-1$$
4. *We have*
$$(g\circ f)(x)=g(f(x))=g(2x-3)=5(2x-3)+1=10x-14.$$
$(g\circ f)(x)$.

5. *If*
$$(f \circ g)(x) = (g \circ f)(x)$$
then we would have
$$10x - 1 = 10x - 14$$
which entails that $-1 = -14$*, absolute nonsense!*

◄

Composition of functions need not be commutative.

178 Example Consider

$$f : \begin{array}{ccc} [-\sqrt{3};\sqrt{3}] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{3-x^2} \end{array}, \quad g : \begin{array}{ccc} [-2;+\infty[& \to & \mathbb{R} \\ x & \mapsto & -\sqrt{x+2} \end{array}.$$

1. Find $\mathbf{Im}\,(f)$.
2. Find $\mathbf{Im}\,(g)$.
3. Find $\mathbf{Dom}\,(f \circ g)$.
4. Find $f \circ g$.
5. Find $\mathbf{Dom}\,(g \circ f)$.
6. Find $g \circ f$.

Solution: ►

1. *Assume* $y = \sqrt{3-x^2}$*. Then* $y \geq 0$*. Moreover* $x = \pm\sqrt{3-y^2}$*. This makes sense only if* $-\sqrt{3} \leq y \leq \sqrt{3}$*. Hence* $\mathbf{Im}\,(f) = [0;\sqrt{3}]$.
2. *Assume* $y = -\sqrt{x+2}$*. Then* $y \leq 0$*. Moreover,* $x = y^2 - 2$ *which makes sense for every real number. This means that* y *is allowed to be any negative number and so* $\mathbf{Im}\,(g) =]-\infty;0]$.
3.
$$\begin{aligned} \mathbf{Dom}\,(f \circ g) &= \{x \in \mathbf{Dom}\,(g) : g(x) \in \mathbf{Dom}\,(f)\} \\ &= \{x \in [-2;+\infty[: -\sqrt{3} \leq -\sqrt{x+2} \leq \sqrt{3}\} \\ &= \{x \in [-2;+\infty[: -\sqrt{3} \leq -\sqrt{x+2} \leq 0\} \\ &= \{x \in [-2;+\infty[: x \leq 1\} \\ &= [-2;1] \end{aligned}$$
4. $(f \circ g)(x) = f(g(x)) = f(-\sqrt{x+2}) = \sqrt{1-x}$.
5.
$$\begin{aligned} \mathbf{Dom}\,(g \circ f) &= \{x \in \mathbf{Dom}\,(f) : f(x) \in \mathbf{Dom}\,(g)\} \\ &= \{x \in [-\sqrt{3};\sqrt{3}] : \sqrt{3-x^2} \geq -2\} \\ &= \{x \in [-\sqrt{3};\sqrt{3}] : \sqrt{3-x^2} \geq 0\} \\ &= [-\sqrt{3};\sqrt{3}] \end{aligned}$$

6. $(g\circ f)(x) = g(f(x)) = g(\sqrt{3-x^2}) = -\sqrt{\sqrt{3-x^2}+2}$.

◀

Notice that $\mathbf{Dom}\,(f\circ g) = [-2;1]$, *although the domain of definition of* $x\mapsto\sqrt{1-x}$ *is* $]-\infty;1]$.

179 Example Let

$$f:\begin{array}{ccc}\mathbb{R}\setminus\{1\} & \to & \mathbb{R}\\ x & \mapsto & \dfrac{2x}{x-1}\end{array},\quad g:\begin{array}{ccc}]-\infty;2] & \to & \mathbb{R}\\ x & \mapsto & \sqrt{2-x}\end{array}$$

1. Find $\mathbf{Im}\,(f)$.
2. Find $\mathbf{Im}\,(g)$.
3. Find $\mathbf{Dom}\,(f\circ g)$.
4. Find $f\circ g$.
5. Find $\mathbf{Dom}\,(g\circ f)$.
6. Find $g\circ f$.

Solution: ▶

1. *Assume* $y=\dfrac{2x}{x-1}$, $x\in\mathbf{Dom}\,(f)$ *is solvable. Then*

$$y(x-1)=2x \implies yx-2x=y \implies x=\frac{y}{y-2}.$$

Thus the equation is solvable only when $y\neq 2$. *Thus* $\mathbf{Im}\,(f)=\mathbb{R}\setminus\{2\}$.

2. *Assume that* $y=\sqrt{2-x}$, $x\in\mathbf{Dom}\,(g)$ *is solvable. Then* $y\geq 0$ *since* $y=\sqrt{2-x}$ *is the square root of a (positive) real number. All* $y\geq 0$ *will render* $x=2-y^2$ *in the appropriate range, and so* $\mathbf{Im}\,(g)=[0;+\infty[$.

3.

$$\begin{aligned}\mathbf{Dom}\,(f\circ g) &= \{x\in\mathbf{Dom}\,(g): g(x)\in\mathbf{Dom}\,(f)\}\\ &= \{x\in]-\infty;2]: \sqrt{2-x}\neq 1\}\\ &=]-\infty;1[\cup]1;2]\end{aligned}$$

4. $(f\circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \dfrac{1}{\sqrt{2-x}-1}$.

5.

$$\begin{aligned}\mathbf{Dom}\,(g\circ f) &= \{x\in\mathbf{Dom}\,(f): f(x)\in\mathbf{Dom}\,(g)\}\\ &= \{x\in\mathbb{R}\setminus\{1\}: \frac{2x}{x-1}\leq 2\}\\ &= \{x\in\mathbb{R}\setminus\{1\}: \frac{2}{x-1}\leq 0\}\\ &=]-\infty;1[\end{aligned}$$

6.

$$(g\circ f)(x) = g(f(x)) = g\left(\frac{2x}{x-1}\right) = \sqrt{2-\frac{2x}{x-1}} = \sqrt{\frac{2}{1-x}}$$

◀

Homework

3.4.1 Problem Let

$$f: \begin{array}{ccc} [-5;3] & \to & \mathbb{R} \\ x & \mapsto & x^4-16 \end{array}, \quad g: \begin{array}{ccc} [-4;2] & \to & \mathbb{R} \\ x & \mapsto & |x|-4 \end{array}.$$

Find

1. **Dom** $(f+g)$
2. **Dom** (fg)
3. **Dom** $\left(\frac{f}{g}\right)$
4. **Dom** $\left(\frac{g}{f}\right)$
5. $(f+g)(2)$
6. $(fg)(2)$
7. $\left(\frac{f}{g}\right)(2)$
8. $\left(\frac{g}{f}\right)(2)$
9. $\left(\frac{f}{g}\right)(1)$
10. $\left(\frac{g}{f}\right)(1)$

3.4.2 Problem Let

$$f: \begin{array}{ccc} \{-2,-1,0,1,2\} & \to & \mathbb{Z} \\ x & \mapsto & 2x \end{array}, \quad g: \begin{array}{ccc} \{0,1,2\} & \to & \mathbb{Z} \\ x & \mapsto & x^2 \end{array}.$$

1. Find **Im** (f).
2. Find **Im** (g).
3. Find **Dom** $(f\circ g)$.
4. Find **Dom** $(g\circ f)$.

3.4.3 Problem Let $f,g,h:\{1,2,3,4\}\to\{1,2,10,1993\}$ be given by

$$f(1)=1,\ f(2)=2,\ f(3)=10,\ f(4)=1993,$$

$$g(1)=g(2)=2, g(3)=g(4)-1=1,$$

$$h(1)=h(2)=h(3)=h(4)+1=2.$$

1. Compute $(f+g+h)(3)$
2. Compute $(fg+gh+hf)(4)$.
3. Compute $f(1+h(3))$.
4. Compute $(f\circ f\circ f\circ f\circ f)(2)+f(g(2)+2)$.

3.4.4 Problem Two functions $f,g:\mathbb{R}\to\mathbb{R}$ are given by $f(x)=ax+b$, $g(x)=bx+a$ with a and b integers. If $f(1)=8$ and $f(g(50))-g(f(50))=28$, find the product ab.

3.4.5 Problem If $a,b,c:\mathbb{R}\to\mathbb{R}$ are functions with $a(t)=t-2, b(t)=t^3, c(t)=5$ demonstrate that

$$\begin{aligned}
(a\circ b)(t) &= t^3-2 \\
(b\circ a)(t) &= (t-2)^3 \\
(b\circ c)(t) &= 125 \\
(c\circ b)(t) &= 5 \\
(c\circ a)(t) &= 5 \\
(a\circ b\circ c)(t) &= 123 \\
(c\circ b\circ a)(t) &= 5 \\
(a\circ c\circ b)(t) &= 3
\end{aligned}$$

3.4.6 Problem Let

$$f: \begin{array}{ccc} [2;+\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x-2} \end{array}, \quad g: \begin{array}{ccc} [-2;2] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{4-x^2} \end{array}.$$

1. Find **Im** (f).
2. Find **Im** (g).
3. Find **Dom** $(f\circ g)$.
4. Find **Dom** $(g\circ f)$.
5. Find $(f\circ g)(x)$.
6. Find $(g\circ f)(x)$.

3.4.7 Problem Let

$$f: \begin{array}{ccc} [-\sqrt{2};+\sqrt{2}[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{2-x^2} \end{array}, \quad g: \begin{array}{ccc}]-\infty;0] & \to & \mathbb{R} \\ x & \mapsto & -\sqrt{-x} \end{array}.$$

1. Find **Im** (f).
2. Find **Im** (g).
3. Find **Dom** $(f\circ g)$.
4. Find **Dom** $(g\circ f)$.
5. Find $(f\circ g)(x)$.
6. Find $(g\circ f)(x)$.

3.4.8 Problem Let $f,g,h:\mathbb{R}\to\mathbb{R}$ be functions. Prove that their composition is associative

$$f\circ(g\circ h)=(f\circ g)\circ h$$

whenever the given expressions make sense.

3.4.9 Problem Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=ax^2-\sqrt{2}$ for some positive a. If $(f\circ f)(\sqrt{2})=-\sqrt{2}$ find the value of a.

3.4.10 Problem Let $f:]0:+\infty[\to]0:+\infty[$, such $f(2x)=\dfrac{2}{2+x}$. Find $2f(x)$.

3.4.11 Problem Let $f,g:\mathbb{R}\setminus\{1\}\to\mathbb{R}$, with $f(x)=\dfrac{4}{x-1}, g(x)=2x$, find all x for which $(g\circ f)(x)=(f\circ g)(x)$.

3.4.12 Problem Let $f:\mathbb{R}\to\mathbb{R}$, $f(1-x)=x^2$. Find $(f\circ f)(x)$.

3.4.13 Problem Let $f:\mathbb{R}\setminus\{-\frac{3}{2}\}\to\mathbb{R}\setminus\{\frac{c}{2}\}, x\mapsto\dfrac{cx}{2x+3}$ be such that $(f\circ f)(x)=x$. Find the value of c.

3.4.14 Problem Let $f,g:\mathbb{R}\to\mathbb{R}$ be functions satisfying for all real numbers x and y the equality

$$f(x+g(y))=2x+y+5. \tag{3.2}$$

Find an expression for $g(x+f(y))$.

3.5 Iteration and Functional Equations

180 Definition Given an assignment rule $x\mapsto f(x)$, its *iterate at x* is $f(f(x))$, that is, we use its value as the new input. The iterates at x

$$x,\ f(x),\ f(f(x)),\ f(f(f(x))),\ldots$$

are called 0-*th iterate*, 1*st iterate*, 2*nd iterate*, 3*rd iterate*, etc. We denote the n-th iterate by $f^{[n]}$.

In some particular cases it is easy to find the nth iterate of an assignment rule, for example

$$a(x)=x^t \implies a^{[n]}(x)=x^{t^n},$$

$$b(x)=mx \implies b^{[n]}(x)=m^nx,$$

$$c(x)=mx+k \implies c^{[n]}(x)=m^nx+k\left(\frac{m^n-1}{m-1}\right).$$

The above examples are more the exception than the rule. Even if its possible to find a closed formula for the n-th iterate some cases prove quite truculent.

181 Example Let $f(x)=\dfrac{1}{1-x}$. Find the n-th iterate of f at x, and determine the set of values of x for which it makes sense.

Solution: ▶ *We have*

$$f^{[2]}(x)=(f\circ f)(x)=f(f(x))=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x},$$

$$f^{[3]}(x)=(f\circ f\circ f)(x)=f(f^{[2]}(x)))=f\left(\frac{x-1}{x}\right)=\frac{1}{1-\frac{x-1}{x}}=x.$$

Notice now that $f^{[4]}(x)=(f\circ f^{[3]})(x)=f(f^{[3]}(x))=f(x)=f^{[1]}(x)$. *We see that f is cyclic of period 3, that is,*

$$f^{[1]}(x)=f^{[4]}(x)=f^{[7]}(x)=\ldots=\frac{1}{1-x},$$

$$f^{[2]}(x)=f^{[5]}(x)=f^{[8]}(x)=\ldots=\frac{x-1}{x},$$

$$f^{[3]}(x)=f^{[6]}(x)=f^{[9]}(x)=\ldots=x.$$

The formulæ above hold for $x\not\in\{0,1\}$. ◀

182 Definition A *functional equation* is an equation whose variables range over functions, or more often, assignment rules.

A functional equation problem asks for a formula, or formulæ satisfying certain features.

183 Example Find all the functions $g:\mathbb{R}\to\mathbb{R}$ satisfying

$$g(x+y)+g(x-y)=2x^2+2y^2.$$

Solution: ► *If* $y = 0$*, then* $2g(x) = 2x^2$*, that is,* $g(x) = x^2$*. Let us verify that* $g(x) = x^2$ *works. We have*

$$g(x+y)+g(x-y) = (x+y)^2+(x-y)^2 = x^2+2xy+y^2+x^2-2xy+y^2 = 2x^2+2y^2,$$

from where the only solution is $g(x) = x^2$*.* ◄

184 Example Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4.$$

Solution: ► *From the given equation,*

$$f(1-x) = 2x - x^4 - x^2 f(x).$$

Replacing x *by* $1-x$*, we obtain*

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4.$$

This implies that

$$f(x) = 2(1-x)-(1-x)^4-(1-x)^2 f(1-x) = 2(1-x)-(1-x)^4-(1-x)^2(2x-x^4-x^2 f(x)),$$

which in turn, gives

$$f(x) = 2(1-x)-(1-x)^4-2x(1-x)^2+x^4(1-x)^2+(1-x)^2x^2 f(x).$$

Solving now for $f(x)$ *we gather that*

$$\begin{aligned}
f(x) &= \frac{2(1-x)-(1-x)^4-2x(1-x)^2+x^4(1-x)^2}{1-(1-x)^2x^2} \\
&= \frac{(1-x)(2-(1-x)^3-2x(1-x)+x^4(1-x)}{)}(1-(1-x)x)(1+(1-x)x) \\
&= \frac{(1-x)(2-(1-3x+3x^2-x^3)-2x+2x^2+x^4-x^5)}{(1-x+x^2)(1+x-x^2)} \\
&= \frac{(1-x)(1+x-x^2+x^3+x^4-x^5)}{(1-x+x^2)(1+x-x^2)} \\
&= \frac{(1-x)(1+x)(1-x+x^2)(1+x-x^2)}{(1-x+x^2)(1+x-x^2)} \\
&= 1-x^2.
\end{aligned}$$

We now check. If $f(x) = 1-x^2$ *then*

$$x^2 f(x) + f(1-x) = x^2(1-x^2)+1-(1-x)^2 = x^2-x^4+1-1+2x-x^2 = 2x-x^4,$$

from $f(x) = 1-x^2$ *is the only such solution.*

◄

We continue with, perhaps, the most famous functional equation.

185 Example (Cauchy's Functional Equation) Suppose $f : \mathbb{Q} \to \mathbb{Q}$ satisfies $f(x+y) = f(x)+f(y)$. Prove that $\exists c \in \mathbb{Q}$ such that $f(x) = cx$, $\forall x \in \mathbb{Q}$.

Solution: ► *Letting $y = 0$ we obtain $f(x) = f(x) + f(0)$, and so $f(0) = 0$. If k is a positive integer we obtain*

$$\begin{aligned}
f(kx) &= f(x+(k-1)x) \\
&= f(x) + f((k-1)x) \\
&= f(x) + f(x) + f((k-2)x) = 2f(x) + f((k-2)x) \\
&= 2f(x) + f(x) + f((k-3)x) = 3f(x) + f((k-3)x) \\
&\vdots \\
&= \cdots = kf(x) + f(0) = kf(x).
\end{aligned}$$

Letting $y = -x$ we obtain $0 = f(0) = f(x) + f(-x)$ and so $f(-x) = -f(x)$. Hence $f(nx) = nf(x)$ for $n \in \mathbb{Z}$. Let $x \in \mathbb{Q}$, which means that $x = \frac{s}{t}$ for integers s, t with $t \neq 0$. This means that $tx = s \cdot 1$ and so $f(tx) = f(s \cdot 1)$ and by what was just proved for integers, $tf(x) = sf(1)$. Hence $f(x) = \frac{s}{t}f(1) = xf(1)$. Since $f(1)$ is a constant, we may put $c = f(1)$. Thus $f(x) = cx$ for rational numbers x. ◄

Homework

3.5.1 Problem Let $f^{[1]}(x) = f(x) = x+1$, $f^{[n+1]} = f \circ f^{[n]}, n \geq 1$. Find a closed formula for $f^{[n]}$

3.5.2 Problem Let $f^{[1]}(x) = f(x) = 2x$, $f^{[n+1]} = f \circ f^{[n]}, n \geq 1$. Find a closed formula for $f^{[n]}$

3.5.3 Problem Find all the assignment rules f that satisfy $f(xy) = yf(x)$.

3.5.4 Problem Find all the assignment rules f for which

$$f(x) + 2f\left(\frac{1}{x}\right) = x.$$

3.5.5 Problem Find all functions $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ such that

$$(f(x))^2 \cdot f\left(\frac{1-x}{1+x}\right) = 64x.$$

3.5.6 Problem An assignment rule f is said to be an *involution* if for all x for which $f(x)$ and $f(f(x))$ are defined we have $f(f(x)) = x$. Prove that $a(x) = \frac{1}{x}$ is an involution for $x \neq 0$.

3.5.7 Problem Prove that $f(x) = \sqrt{1-x^2}$ is an involution for $0 \leq x \leq 1$.

3.5.8 Problem Let f satisfy $f(n+1) = (-1)^{n+1}n - 2f(n), n \geq 1$ If $f(1) = f(1001)$ find $f(1) + f(2) + f(3) + \cdots + f(1000)$.

3.5.9 Problem Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(1) = 1, \quad \forall x \in \mathbb{R} \quad f(x+3) \geq f(x) + 3, \quad f(x+1) \leq f(x) + 1.$$

Put $g(x) = f(x) - x + 1$. Determine $g(2008)$.

3.5.10 Problem If $f(a)f(b) = f(a+b)\ \forall\ a, b \in \mathbb{R}$ and $f(x) > 0\ \forall\, x \in \mathbb{R}$, find $f(0)$. Also, find $f(-a)$ and $f(2a)$ in terms of $f(a)$.

3.6 Injections and Surjections

186 Definition A function

$$f : \begin{array}{ccc} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ a & \mapsto & f(a) \end{array}$$

is said to be *injective* or *one-to-one* if $(a_1, a_2) \in (\mathbf{Dom}(f))^2$,

$$a_1 \neq a_2 \implies f(a_1) \neq f(a_2).$$

That is,

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

f is said to be *surjective* or *onto* if $\mathbf{Target}(f) = \mathbf{Im}(f)$. That is, if $(\forall b \in B)$ $(\exists a \in A)$ such that $f(a) = b$. f is *bijective* if it is both injective and surjective. The number a is said to the the *pre-image* of b.

A function is thus injective if different inputs result in different outputs, and it is surjective if every element of the target set is hit. Figures 3.18 through 3.21 present various examples.

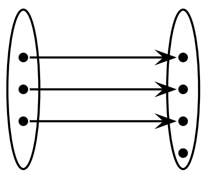

Figure 3.18: Injective, not surjective.

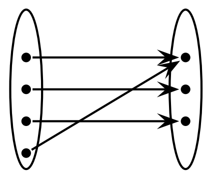

Figure 3.19: Surjective, not injective.

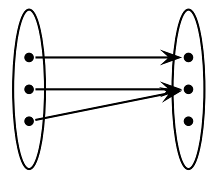

Figure 3.20: Neither injective nor surjective.

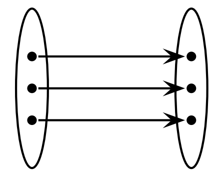

Figure 3.21: Bijective.

It is apparent from figures 3.18 through 3.21 that if the domain and the target set of a function are finite, then there are certain inequalities that must be met in order for the function to be injective, surjective or bijective. We make the precise statement in the following theorem.

187 Theorem Let $f : A \to B$ be a function, and let A and B be finite, with A having n elements, and and B m elements. If f is injective, then $n \leq m$. If f is surjective then $m \leq n$. If f is bijective, then $m = n$. If $n \leq m$, then the number of injections from A to B is

$$m(m-1)(m-2)\cdots(m-n+1).$$

Proof: *Let* $A = \{x_1, x_2, \ldots, x_n\}$ *and* $B = \{y_1, y_2, \ldots, y_m\}$.

If f were injective then $f(x_1), f(x_2), \ldots, f(x_n)$ are all distinct, and among the y_k. Hence $n \leq m$. In this case, there are m choices for $f(x_1)$, $m-1$ choices for $f(x_2)$, $\ldots$, $m-n+1$ choices for $f(x_n)$. Thus there are

$$m(m-1)(m-2)\cdots(m-n+1)$$

injections from A to B.

If f were surjective then each y_k is hit, and for each, there is an x_i with $f(x_i) = y_k$. Thus there are at least m different images, and so $n \geq m$. ❑

To find the number of surjections from a finite set to a finite set we need to know about Stirling numbers and inclusion-exclusion, and hence, we refer the reader to any good book in Combinatorics.

188 Example Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$. How many functions are there from A to B? How many functions are there from B to A? How many injections are there from A to B? How many surjections are there from B to A?

Solution: ► *There are $4 \cdot 4 \cdot 4 = 64$ functions from A to B, since there are 4 possibilities for the image of 1, 4 for the image of 2, and 4 for the image of 3. Similarly, there are $3 \cdot 3 \cdot 3 \cdot 3 = 81$ functions from B to A.*

By Theorem 187, there are

$$4 \cdot 3 \cdot 2 = 24$$

injections from A to B.

The 3^4 functions from B to A come in three flavours: (i) those that are surjective, (ii) those that map to exactly two elements of A, and (iii) those that map to exactly one element of A.

Take a particular element of A, say $1 \in A$. There are 2^4 functions from B to $\{2,3\}$. Notice that some of these may map to the whole set $\{2,3\}$ or they may skip an element. Coupling this with the $1 \in A$, this means that there are 2^4 functions from B to A that skip the 1 and may or may not skip the 2 or the 3. Since there is nothing holy about choosing $1 \in A$, we conclude that there are $3 \cdot 2^4$ from B to A that skip either one or two elements of A.

Now take two particular elements of A, say $\{1,2\} \subseteq A$. There are 1^4 functions from B to $\{3\}$. Since there are three 2-element subsets in A—namely $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$—this means that there are $3 \cdot 1^4$ functions from B to A that map precisely into one element of A.

To find the number of surjections from B to A we weed out the functions that skip elements. In considering the difference $3^4 - 3 \cdot 2^4$, we have taken out all the functions that miss one or two elements of A, but in so doing, we have taken out twice those that miss one element. Hence we put those back in and we obtain

$$3^4 - 3 \cdot 2^4 + 3 \cdot 1^4 = 36$$

surjections from B to A. ◄

It is easy to see that a graphical criterion for a function to be injective is that every horizontal line crossing the function must meet it at most one point. See figures 3.22 and 3.23.

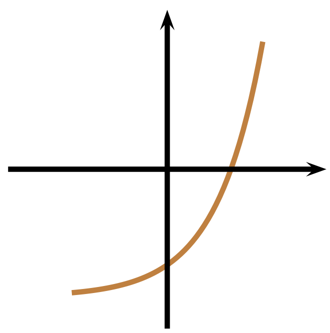

Figure 3.22: Passes horizontal line test: injective.

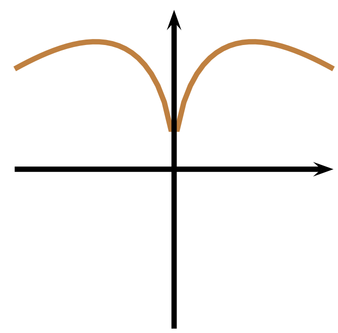

Figure 3.23: Fails horizontal line test: not- injective.

189 Example The $a : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$ is neither injective nor surjective. For example, $a(-2) = a(2) = 4$ but $-2 \neq 2$, and there is no $x \in \mathbb{R}$ with $a(x) = -1$. The function $b : \begin{array}{ccc} \mathbb{R} & \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}$ is surjective but not injective. The function $c : \begin{array}{ccc} [0;+\infty[& \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$ is injective but not surjective. The function $d : \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}$ is bijective.

Given a formula, it is particularly difficult to know in advance what it set of outputs is going to be. This is why when we talk about function, we specify the target set to be a canister for every possible value. The next few examples shew how to find the image of a formula in a few easy cases.

190 Example Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x + 3$. Determine $\mathbf{Im}(f)$.

Solution: ► *Observe that*

$$x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x+1)^2 + 2 \geq 2,$$

since the square of every real number is positive. Since $(x+1)^2$ *could be made as arbitrarily close to* 0 *as desired (upon taking values of* x *close to* -1*), and can also be made as large as desired, we conclude that* $\mathbf{Im}(f) \subseteqq [2;+\infty[$*. Now, let* $a \in [2;+\infty[$*. Then*

$$x^2+2x+3=a \iff (x+1)^2+2=a \iff x=-1\pm\sqrt{a-2}.$$

Since $a \geq 2$, $\sqrt{a-2} \in \mathbb{R}$ *and* $x \in \mathbb{R}$*. This means that* $[2;+\infty[\subseteqq \mathbf{Im}(f)$ *and so we conclude that* $\mathbf{Im}(f) = [2:+\infty[$*.*
◀

191 Example Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f(x) = \dfrac{2x}{x-1}$. Determine $\mathbf{Im}(f)$.

Solution: ▶ *Observe that*

$$\frac{2x}{x-1} = 2 + \frac{2}{x-1} \neq 2$$

since $\dfrac{2}{x-1}$ *never vanishes for any real number* x*. We will shew that* $\mathbf{Im}(f) = \mathbb{R} \setminus \{2\}$*. For let* $a \neq 2$*. Then*

$$\frac{2x}{x-1} = a \implies 2x = ax - a \implies x(2-a) = -a \implies x = \frac{a}{a-2}.$$

But if $a \neq 2$*, then* $x \in \mathbb{R}$ *and so we conclude that* $\mathbf{Im}(f) = \mathbb{R} \setminus \{2\}$*.* ◀

192 Example Consider the function $f : \begin{array}{ccc} x & \to & \dfrac{x-1}{x+1} \\ A & \mapsto & B \end{array}$, where A is the domain of definition of f.

1. Determine A.
2. Determine B so that f be surjective.
3. Demonstrate that f is injective.

Solution: ▶ *The formula* $f(x) = \dfrac{x-1}{x+1}$ *outputs real numbers for all values of* x *except for* $x = -1$*, whence* $A = \mathbb{R} \setminus \{-1\}$*.*
Now,

$$\frac{x-1}{x+1} = 1 + \frac{2}{x-1} \neq 1,$$

since $\dfrac{2}{x-1}$ *never vanishes. If* $a \neq 1$ *then*

$$\frac{x-1}{x+1} = a \implies ax - a = x+1 \implies x(a-1) = 1+a \implies x = \frac{1+a}{1-a},$$

which is a real number, since $a \neq 1$*. It follows that* $\mathbf{Im}(f) = \mathbb{R} \setminus \{1\}$*.*
To demonstrate that f *is injective, we observe that*

$$f(a) = f(b) \implies \frac{a-1}{a+1} = \frac{b-1}{b+1} \implies (a-1)(b+1) = (a+1)(b-1) \implies ab+a-b = ab-a+b \implies 2a = 2b \implies a = b,$$

from where the function is indeed injective.
◀

193 Example Prove that

$$h : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is a bijection.

Solution: ► *Assume $h(b) = h(a)$. Then*

$$\begin{aligned} h(a) = h(b) &\implies a^3 = b^3 \\ &\implies a^3 - b^3 = 0 \\ &\implies (a-b)(a^2+ab+b^2) = 0 \end{aligned}$$

Now,

$$b^2 + ab + a^2 = \left(b + \frac{a}{2}\right)^2 + \frac{3a^2}{4}.$$

This shews that $b^2 + ab + a^2$ is positive unless both a and b are zero. Hence $b - a = 0$ in all cases. We have shewn that $h(b) = h(a) \implies b = a$, and the function is thus injective.

To prove that h is surjective, we must prove that $(\forall\, b \in \mathbb{R})\ (\exists a)$ such that $h(a) = b$. We choose a so that $a = b^{1/3}$. Then

$$h(a) = h(b^{1/3}) = (b^{1/3})^3 = b.$$

Our choice of a works and hence the function is surjective. ◄

194 Example Prove that $f : \begin{array}{ccc} \mathbb{R} \setminus \{1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{x^{1/3}}{x^{1/3}-1} \end{array}$ is injective but not surjective.

Solution: ► *We have*

$$\begin{aligned} f(a) = f(b) &\implies & \frac{a^{1/3}}{a^{1/3}-1} &= \frac{b^{1/3}}{b^{1/3}-1} \\ &\implies & a^{1/3}b^{1/3} - a^{1/3} &= a^{1/3}b^{1/3} - b^{1/3} \\ &\implies & -a^{1/3} &= -b^{1/3} \\ &\implies & a &= b, \end{aligned}$$

whence f is injective. To prove that f is not surjective assume that $f(x) = b, b \in \mathbb{R}$. Then

$$f(x) = b \implies \frac{x^{1/3}}{x^{1/3}-1} = b \implies x = \frac{b^3}{(b-1)^3}.$$

The expression for x is not a real number when $b = 1$, and so there is no real x such that $f(x) = 1$. ◄

195 Example Find the image of the function

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{x-1}{x^2+1} \end{array}$$

Solution: ► *First observe that $f(x) = 0$ has the solution $x = 1$. Assume $b \in \mathbb{R}$, $b \neq 0$, with $f(x) = b$. Then*

$$\frac{x-1}{x^2+1} = b \implies bx^2 - x + b + 1 = 0.$$

Completing squares,

$$bx^2 - x + b + 1 = b\left(x^2 - \frac{x}{b}\right) + b + 1 = b\left(x^2 - \frac{x}{b} + \frac{1}{4b^2}\right) + b + 1 - \frac{1}{4b} = b\left(x - \frac{1}{2b}\right)^2 + \frac{-1+4b+4b^2}{4b}.$$

Hence

$$bx^2 - x + b + 1 = 0 \iff b\left(x - \frac{1}{2b}\right)^2 = \frac{1-4b-4b^2}{4b} \iff x = \frac{1}{2} \pm \frac{\sqrt{1-4b-4b^2}}{2b}.$$

We must in turn investigate the values of b for which $b \neq 0$ *and* $1-4b-4b^2 \geq 0$. *Again, completing squares*

$$1-4b-4b^2 = -4\left(b^2+b\right)+1 = -4\left(b^2+b+\frac{1}{4}\right)+2 = 2-(2b+1)^2 == \left(\sqrt{2}-2b-1\right)\left(\sqrt{2}+2b+1\right).$$

A sign diagram then shews that $1-4b-4b^2 \geq 0$ *for*

$$b \in \left[-\frac{1}{2}-\frac{\sqrt{2}}{2}; -\frac{1}{2}+\frac{\sqrt{2}}{2}\right],$$

and so

$$\mathbf{Im}\,(f) = \left[-\frac{1}{2}-\frac{\sqrt{2}}{2}; -\frac{1}{2}+\frac{\sqrt{2}}{2}\right].$$

◀

Homework

3.6.1 Problem Prove that

$$g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ s & \mapsto & 2s+1 \end{array}$$

is a bijection.

3.6.2 Problem Prove that $h : \mathbb{R} \to \mathbb{R}$ given by $h(s) = 3 - s$ is a bijection.

3.6.3 Problem Prove that $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^{1/3}$ is a bijection.

3.6.4 Problem Prove that $f : \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R}\setminus\{2\} \\ x & \mapsto & \dfrac{2x}{x+1} \end{array}$ is surjective but that $g : \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{2x}{x+1} \end{array}$ is not surjective.

3.6.5 Problem Classify each of the following as injective, surjective, bijective or neither.

1. $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^4$
2. $f : \mathbb{R} \to \{1\}, \quad x \mapsto 1$
3. $f : \{1,2,3\} \to \{a,b\}, \quad f(1) = f(2) = a, f(3) = b$
4. $f : [0;+\infty[\to \mathbb{R}, \quad x \mapsto x^3$
5. $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto |x|$
6. $f : [0;+\infty[\to \mathbb{R}, \quad x \mapsto -|x|$
7. $f : \mathbb{R} \to [0;+\infty[, \quad x \mapsto |x|$
8. $f : [0;+\infty[\to [0;+\infty[, \quad x \mapsto x^4$

3.6.6 Problem Let $f : E \to F, g : F \to G$ be two functions. Prove that if $g \circ f$ is surjective then g is surjective.

3.6.7 Problem Let $f : E \to F, g : F \to G$ be two functions. Prove that if $g \circ f$ is injective then f is injective.

3.7 Inversion

Let $S \subsetneqq \mathbb{R}$. Recall that $\mathbf{Id}_S$ is the identity function on S, that is , $\mathbf{Id}_S : S \to S$ with$\mathbf{Id}_S(x) = x$.

196 Definition Let $A \times B \subseteq \mathbb{R}^2$. A function $f : A \to B$ is said to be *right invertible* if there is a function $g : B \to A$, called the *right inverse of* f such that $f \circ g = \mathbf{Id}_B$. In the same fashion, f is said to be *left invertible* if there exists a function $h : B \to A$ such that $h \circ f = \mathbf{Id}_A$. A function is *invertible* if it is both right and left invertible.

197 Theorem Let $f : A \to B$ be right and left invertible. Then its left inverse coincides with its right inverse.

Proof: *Let $g, h : B \to A$ be the respective right and left inverses of f. Using the associativity of compositions,*

$$(f \circ g) = (\mathbf{Id}_B) \implies h \circ (f \circ g) = h \circ \mathbf{Id}_B \implies (h \circ f) \circ g = h \circ \mathbf{Id}_B \implies (\mathbf{Id}_A) \circ g = h \circ \mathbf{Id}_B \implies g = h.$$

❑

198 Corollary (Uniqueness of Inverses) If $f : A \to B$ is invertible, then its inverse is unique.

Proof: *Let f have the two inverses $s, t : B \to A$. In particular, s would be a right inverse and t would be a left inverse. By the preceding theorem, these two must coincide.* ❑

199 Definition If $f : A \to B$ is invertible, then its inverse will be denoted by $f^{-1} : B \to A$.

We must alert the reader that f^{-1} does not denote the reciprocal (multiplicative inverse) of f.

200 Theorem Let $f : A \to B$ and $g : C \to A$ be invertible. Then the composition function $f \circ g : C \to B$ is also invertible and

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

Proof: *By the uniqueness of inverses, $f \circ g$ may only have one inverse, which is, by definition, $(f \circ g)^{-1}$. This means that any other function that satisfies the conditions of being an inverse of $f \circ g$ must then by default be* the *inverse of $f \circ g$. We have,*

$$(g^{-1} \circ f^{-1}) \circ (f \circ g) = g^{-1} \circ (f^{-1} \circ f) \circ g = g^{-1} \circ \mathbf{Id}_A \circ g = g^{-1} \circ g = \mathbf{Id}_C.$$

In the same fashion,

$$(f \circ g) \circ (g^{-1} \circ f^{-1}) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ \mathbf{Id}_A \circ f^{-1} = f \circ f^{-1} = \mathbf{Id}_B.$$

The theorem now follows from the uniqueness of inverses.❑

201 Example Let $f : \begin{array}{ccc} \mathbb{R} \setminus \{1\} & \to & \mathbb{R} \setminus \{2\} \\ x & \mapsto & \dfrac{2x}{x-1} \end{array}$. Demonstrate that $g : \begin{array}{ccc} \mathbb{R} \setminus \{2\} & \to & \mathbb{R} \setminus \{1\} \\ x & \mapsto & \dfrac{x}{x-2} \end{array}$ is the inverse of f.

Solution: ► *Let $x \in \mathbb{R} \setminus \{2\}$. We have*

$$(f \circ g)(x) = f(g(x)) = \frac{2g(x)}{g(x)-1} = \frac{\dfrac{2x}{x-2}}{\dfrac{x}{x-2} - 1} = \frac{2x}{x-(x-2)} = x,$$

from where g is a right inverse of f. In a similar manner, $x \in \mathbb{R} \setminus \{2\}$,

$$(g \circ f)(x) = g(f(x)) = \frac{f(x)}{f(x)-2} = \frac{\dfrac{2x}{x-1}}{\dfrac{2x}{x-1} - 2} = \frac{2x}{2x-2(x-1)} = x,$$

whence g is a left inverse of f. ◄

Consider the functions $u : \{a, b, c\} \to \{x, y, z\}$ and $v : \{x, y, z\} \to \{a, b, c\}$ as given by diagram 3.24. It is clear the v undoes whatever u does. Furthermore, we observe that u and v are bijections and that the domain of u is the image of v and vice-versa. This example motivates the following theorem.

202 Theorem A function $f : A \to B$ is invertible if and only if it is a bijection.

Proof: *Assume first that f is invertible. Then there is a function $f^{-1} : B \to A$ such that*

$$f \circ f^{-1} = \mathbf{Id}_B \ \text{ and } \ f^{-1} \circ f = \mathbf{Id}_A. \tag{3.3}$$

Let us prove that f is injective and surjective. Let s, t be in the domain of f and such that $f(s) = f(t)$. Applying f^{-1} to both sides of this equality we get $(f^{-1} \circ f)(s) = (f^{-1} \circ f)(t)$. By the definition of inverse function, $(f^{-1} \circ f)(s) = s$ and $(f^{-1} \circ f)(t) = t$. Thus $s = t$. Hence $f(s) = f(t) \implies s = t$ implying that f is injective. To prove that f is surjective we must shew that for every $b \in f(A)$ $\exists a \in A$ such that $f(a) = b$. We take $a = f^{-1}(b)$ (observe that $f^{-1}(b) \in A$). Then $f(a) = f(f^{-1}(b)) = (f \circ f^{-1})(b) = b$ by definition of inverse function. This shews that f is surjective. We conclude that if f is invertible then it is also a bijection.

Assume now that f is a bijection. For every $b \in B$ there exists a unique a such that $f(a) = b$. This makes the rule $g : B \to A$ given by $g(b) = a$ a function. It is clear that $g \circ f = \mathbf{Id}_A$ and $f \circ g = \mathbf{Id}_B$. We may thus take $f^{-1} = g$. This concludes the proof. ❑

Figure 3.24: A function and its inverse.

We will now give a few examples of how to determine the assignment rule of the inverse of a function.

203 Example Assume that the function

$$f : \begin{array}{ccc} \mathbb{R} \setminus \{-1\} & \to & \mathbb{R} \setminus \{1\} \\ x & \mapsto & \dfrac{x-1}{x+1} \end{array}$$

is a bijection. Determine its inverse.

Solution: ► *Put*

$$\frac{x-1}{x+1} = y$$

and solve for x:

$$\frac{x-1}{x+1} = y \implies x - 1 = yx + y \implies x - yx = 1 + y \implies x(1-y) = 1 + y \implies x = \frac{1+y}{1-y}.$$

Now, exchange x and y: $y = \dfrac{1+x}{1-x}$. The desired inverse is

$$f^{-1} : \begin{array}{ccc} \mathbb{R} \setminus \{1\} & \to & \mathbb{R} \setminus \{-1\} \\ x & \mapsto & \dfrac{1+x}{1-x} \end{array}.$$

◄

204 Example Assume that the function

$$f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (x-2)^3+1 \end{array}$$

is a bijection. Determine its inverse.

Solution: ► *Put*

$$(x-2)^3+1=y$$

and solve for x:

$$(x-2)^3+1=y \implies (x-2)^3=y-1 \implies x-2=\sqrt[3]{y-1} \implies x=\sqrt[3]{y-1}+2.$$

Now, exchange x and y: $y=\sqrt[3]{x-1}+2$. *The desired inverse is*

$$f^{-1}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \sqrt[3]{x-1}+2 \end{array}.$$

◄

Since by Theorem 107, $(x, f(x))$ and $(f(x), x)$ are symmetric with respect to the line $y=x$, the graph of a function f is symmetric with its inverse with respect to the line $y=x$. See figures 3.25 through 3.27.

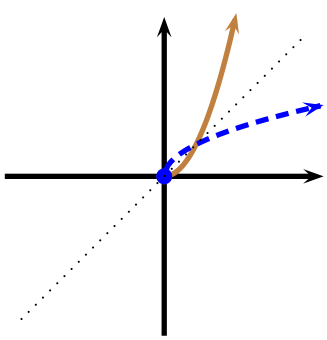

Figure 3.25: Function and its inverse.

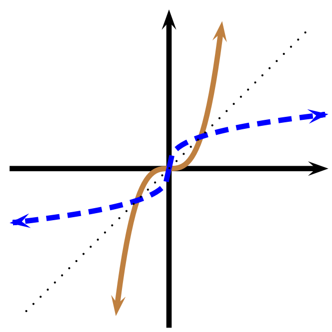

Figure 3.26: Function and its inverse.

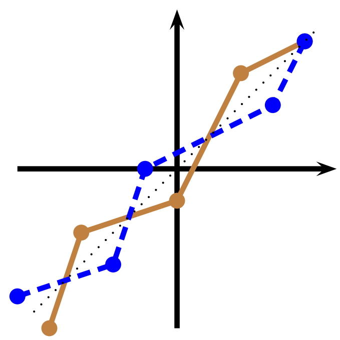

Figure 3.27: Function and its inverse.

205 Example Consider the functional curve in figure 3.28.

1. Determine $\mathbf{Dom}(f)$.
2. Determine $\mathbf{Im}(f)$.
3. Draw the graph of f^{-1}.
4. Determine $f(+5)$.
5. Determine $f^{-1}(-2)$.
6. Determine $f^{-1}(-1)$.

Solution: ►

1. $[-5;5]$

2. $[-3;3]$

3. *To obtain the graph, we look at the endpoints of lines on the graph of f and exchange their coordinates. Thus the endpoints $(-5,-3)$, $(-3,-2)$, $(0,-1)$, $(1,1)$, $(5,3)$ on the graph of f now form the endpoints $(-3,-5)$, $(-2,-3)$, $(-1,0)$, $(1,1)$, and $(3,5)$ on the graph of f^{-1}. The graph appears in figure 3.29 below.*

4. $f(+5) = 3$.
5. $f^{-1}(-2) = -3$.
6. $f^{-1}(-1) = 0$.

◄

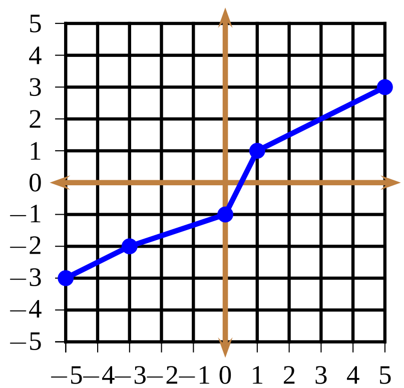

Figure 3.28: f for example 205.

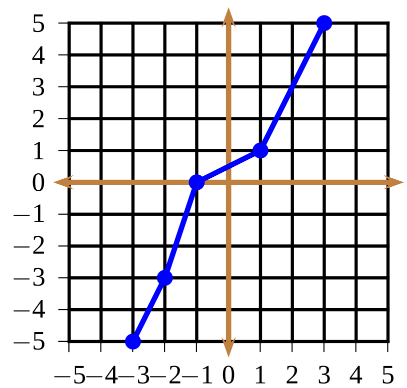

Figure 3.29: f^{-1} for example 205.

206 Example Consider the formula $f(x) = x^2+4x+5$. Demonstrate that f is injective in $[-2;+\infty[$ and determine $f([-2;+\infty[)$. Then, find the inverse of

$$f: \begin{array}{ccc} [-2;+\infty[& \to & f([-2;+\infty[) \\ x & \mapsto & x^2+4x+5 \end{array}.$$

Solution: ► *Observe that $x^2+4x+5 = (x+2)^2+1$. Now, if $a \in [-2;+\infty[$ and $b \in [-2;+\infty[$, then*

$$f(a) = f(b) \implies (a+2)^2+1 = (b+2)^2+1 \implies (a+2)^2 = (b+2)^2.$$

As $a+2 \geq 0$ and $b+2 \geq 0$, we have

$$(a+2)^2 = (b+2)^2 \implies a+2 = b+2 \implies a = b,$$

whence f is injective in $[-2;+\infty[$.

We have $f(x) = (x+2)^2+1 \geq 1$. We will shew that $f([-2;+\infty[= [1;+\infty[$. Let $b \in [1;+\infty[$. Solving for x:

$$f(x) = b \implies (x+2)^2+1 = b \implies (x+2)^2 = b-1.$$

As $b-1 \geq 0$, $\sqrt{b-1}$ is a real number and thus

$$x = -2+\sqrt{b-1}$$

is a real number with $x \leq -2$. We deduce that $f([-2;+\infty[) = [1;+\infty[$.

Since

$$f: \begin{array}{ccc} [-2;+\infty[& \to & [1;+\infty[\\ x & \mapsto & x^2+4x+5 \end{array}$$

is a bijection, it is invertible. To find f^{-1}, we solve

$$x^2+4x+5=y \implies (x+2)^2+1=y \implies x=-2+\sqrt{y-1},$$

where we have taken the positive square root, since $x \geq -2$. Exchanging x and y we obtain $y=-2+\sqrt{x-1}$. We deduce that the inverse of f is

$$f^{-1}: \begin{array}{ccc} [1;+\infty[& \to & [-2;+\infty[\\ x & \mapsto & -2+\sqrt{x-1} \end{array}.$$

◀

In the same fashion it is possible to demonstrate that

$$g: \begin{array}{ccc}]-\infty;-2] & \to & [1;+\infty[\\ x & \mapsto & x^2+4x+5 \end{array}$$

bijective is, with inverse

$$g^{-1}: \begin{array}{ccc} [1;+\infty[& \to &]-\infty;-2] \\ x & \mapsto & -2-\sqrt{x-1} \end{array}.$$

Homework

3.7.1 Problem Let

$$c: \begin{array}{ccc} \mathbb{R}\setminus\{-2\} & \to & \mathbb{R}\setminus\{1\} \\ x & \mapsto & \dfrac{x}{x+2} \end{array}.$$

Prove that c is bijective and find the inverse of c.

3.7.2 Problem Assume that $f:\mathbb{R}\to\mathbb{R}$ is a bijection, where $f(x)=2x^3+1$. Find $f^{-1}(x)$.

3.7.3 Problem Assume that $f:\mathbb{R}\setminus\{1\}\to\mathbb{R}\setminus\{1\}$ is a bijection, where $f(x)=\sqrt[3]{\dfrac{x+2}{x-1}}$. Find f^{-1}.

3.7.4 Problem Let f and g be invertible functions satisfying

$$f(1)=2, \quad f(2)=3, \quad f(3)=1,$$

$$g(1)=-1, \quad g(2)=3, \quad g(4)=-2.$$

Find $(f\circ g)^{-1}(1)$.

3.7.5 Problem Consider the formula $f: x\mapsto x^2-4x+5$. Find two intervals I_1 and I_2 with $\mathbb{R}=I_1\cup I_2$ and $I_1\cap I_2$ consisting on exactly one point, such that f be injective on the restrictions to each interval $f\big|_{I_1}$ and $f\big|_{I_2}$. Then, find the inverse of f on each restriction.

3.7.6 Problem Consider the function $f:[-5;5]\to[-3;5]$ whose graph appears in figure 3.30, and which is composed of two lines. Observe that f passes the horizontal line test, that it is surjective, and hence invertible. .

1. Find a formula for f and f^{-1} in $[-5;0]$.
2. Find a formula for f and f^{-1} in $[0;5]$.
3. Draw the graph of f^{-1}.

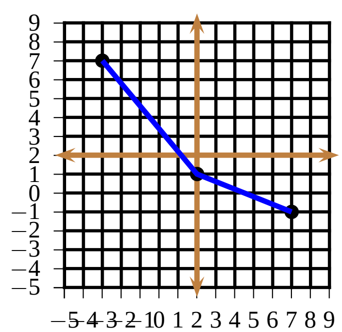

Figure 3.30: Problem 3.7.6.

3.7.7 Problem Consider the rule

$$f(x)=\frac{1}{\sqrt[3]{x^5-1}}.$$

1. Find the natural domain of f.
2. Find the inverse assignment rule f^{-1}.
3. Find the image of the natural domain of f and the natural domain of f^{-1}.
4. Conclude.

3.7.8 Problem Find all the real solutions to the equation

$$x^2 - \frac{1}{4} = \sqrt{x + \frac{1}{4}}.$$

3.7.9 Problem Let $f, g, h : \{1,2,3,4\} \to \{1,2,10,1993\}$ be given by $f(1) = 1, f(2) = 2, f(3) = 10, f(4) = 1993, g(1) = g(2) = 2, g(3) = g(4) - 1 = 1, h(1) = h(2) = h(3) = h(4) + 1 = 2.$

1. Is f invertible? Why? If so, what is $f^{-1}(f(h(4)))$?
2. Is g one-to-one? Why?

3.7.10 Problem Given $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + 8$ and $f : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{0\}$, $f(x) = \dfrac{1}{x+2}$ find $(g \circ f^{-1})(-2)$.

3.7.11 Problem Prove that $t : \begin{array}{ccc}]-\infty;1] & \to & [0;+\infty[\\ x & \mapsto & \sqrt{1-x} \end{array}$ is a bijection and find t^{-1}.

3.7.12 Problem Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$. For which parameters a and b is $f = f^{-1}$?

3.7.13 Problem Prove that if $ab \neq -4$ and $f : \mathbb{R} \setminus \{2/b\} \to \mathbb{R} \setminus \{2/b\}$, $f(x) = \dfrac{2x+a}{bx-2}$ then $f = f^{-1}$.

3.7.14 Problem Let $f : [0;+\infty[\to [0;+\infty[$ be given by

$$f(x) = \sqrt{x + \sqrt{x}}.$$

Demonstrate that f is bijective and that its inverse is

$$f^{-1} : [0;+\infty[\to [0;+\infty[, \quad f^{-1}(x) = \frac{1 - \sqrt{1+4x^2}}{2} + x^2.$$

3.7.15 Problem Demonstrate that

$$f : \mathbb{R} \to [-1;1], \quad f(x) = \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt[3]{1+x} + \sqrt[3]{1-x}},$$

is bijective and that its inverse is

$$f^{-1} : [-1;1] \to \mathbb{R}, \quad f^{-1}(x) = \frac{x(x^2+3)}{1+3x^2}.$$

3.7.16 Problem Demonstrate that

$$f : \left[-\frac{1}{4};+\infty\right[\to]-1;1], \quad f(x) = \frac{1 - \sqrt{1+4x}}{1 + \sqrt{1+4x}},$$

is bijective and that its inverse is

$$f^{-1} :]-1;1] \to \left[-\frac{1}{4};+\infty\right[, \quad f^{-1}(x) = -\frac{x}{(1+x)^2}.$$

3.7.17 Problem Demonstrate that

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt[3]{x + \sqrt{x^2+1}} + \sqrt[3]{x - \sqrt{x^2+1}},$$

is bijective and that its inverse is

$$f^{-1} : \mathbb{R} \to \mathbb{R}, \quad f^{-1}(x) = \frac{x^3+3x}{2}.$$

3.7.18 Problem Consider the function $f : \mathbb{R} \to \mathbb{R}$, with

$$f(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

whose graph appears in figure 3.31.

1. Is f invertible?
2. If the previous answer is affirmative, draw the graph of f^{-1}.
3. If f is invertible, find a formula for f^{-1}.

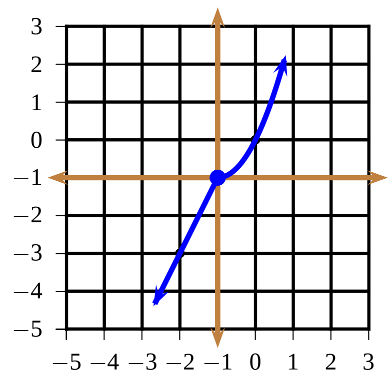

Figure 3.31: Problem 3.7.18.

3.7.19 Problem Demonstrate that $f : [0;1] \to [0;1]$, with

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0;1] \\ 1-x & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0;1] \end{cases}$$

is bijective and that $f = f^{-1}$.

3.7.20 Problem Prove, without using a calculator, that

$$\sum_{k=1}^{9} \left(\left(\frac{k}{10} \right)^2 + \sqrt{\frac{k}{10}} \right) < 9.5$$

3.7.21 Problem Verify that the functions below, with their domains and images, have the claimed inverses.

Assignment Rule	Natural Domain	Image	Inverse
$x \mapsto \sqrt{2-x}$	$]-\infty;2]$	$[0;+\infty[$	$x \mapsto 2 - x^2$
$x \mapsto \dfrac{1}{\sqrt{2-x}}$	$]-\infty;2[$	$]0;+\infty[$	$x \mapsto 2 - \dfrac{1}{x^2}$
$x \mapsto \dfrac{2+x^3}{2-x^3}$	$\mathbb{R} \setminus \{\sqrt[3]{2}\}$	$\mathbb{R} \setminus \{-1\}$	$x \mapsto \sqrt[3]{\dfrac{2x-2}{x+1}}$
$x \mapsto \dfrac{1}{x^3-1}$	$\mathbb{R} \setminus \{1\}$	$\mathbb{R} \setminus \{0\}$	$x \mapsto \sqrt[3]{1 + \dfrac{1}{x}}$

4 Transformations of the Graph of Functions

4.1 Translations

In this section we study how several rigid transformations affect both the graph of a function and its assignment rule.

207 Theorem Let f be a function and let v and h be real numbers. If (x_0, y_0) is on the graph of f, then $(x_0, y_0 + v)$ is on the graph of g, where $g(x) = f(x) + v$, and if (x_1, y_1) is on the graph of f, then $(x_1 - h, y_1)$ is on the graph of j, where $j(x) = f(x+h)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff y_0 + v = f(x_0) + v \iff y_0 + v = g(x_0) \iff (x_0, y_0 + v) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f(x_1 - h + h) \iff y_1 = j(x_1 - h) \iff (x_1 - h, y_1) \in \Gamma_j.$$

❑

208 Definition Let f be a function and let v and h be real numbers. We say that the curve $y = f(x) + v$ is a *vertical translation* of the curve $y = f(x)$. If $v > 0$ the translation is v up, and if $v < 0$, it is v units down. Similarly, we say that the curve $y = f(x+h)$ is a *horizontal translation* of the curve $y = f(x)$. If $h > 0$, the translation is h units left, and if $h < 0$, then the translation is h units right.

Given a functional curve, we expect that a translation would somehow affect its domain and image.

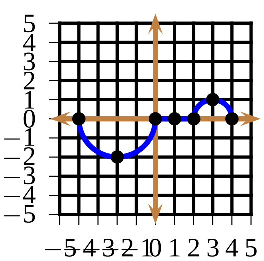

Figure 4.1: $y = f(x)$.

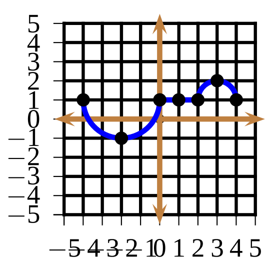

Figure 4.2: $y = f(x) + 1$.

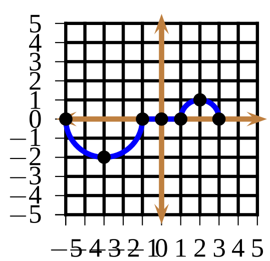

Figure 4.3: $y = f(x+1)$.

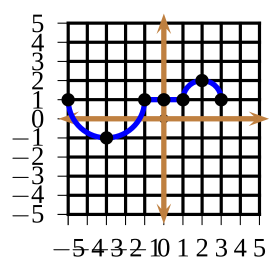

Figure 4.4: $y = f(x+1) + 1$.

209 Example Figures 4.2 through 4.4 shew various translations of $f : [-4;4] \to [-2;1]$ in figure 4.1. Its translation $a : [-4;4] \to [-1;2]$ one unit up is shewn in figure 4.2. Notice that we have simply increased the y-coordinate of every point on the original graph by 1, without changing the x-coordinates. Its translation $b : [-5;3] \to [-2;1]$ one unit left is shewn in figure 4.3. Its translation $c : [-5;3] \to [-1;2]$ one unit up and one unit left is shewn in figure 4.4. Notice how the domain and image of the original curve are affected by the various translations.

210 Example Consider

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}.$$

Figures 4.5, 4.6 and 4.7 shew the vertical translation a 3 units up and the vertical translation b 3 units down, respectively. Observe that

$$a : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2+3 \end{array}, \qquad b : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2-3 \end{array}.$$

Figures 4.8 and 4.9, respectively shew the horizontal translation c 3 units right, and the horizontal translation d 3 units left. Observe that

$$c : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (x-3)^2 \end{array}, \qquad d : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (x+3)^2 \end{array}.$$

Figure 4.10, shews g, the simultaneous translation 3 units left and down. Observe that

$$g : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (x+3)^3-3 \end{array}.$$

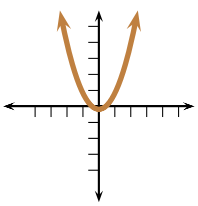
Figure 4.5: $y = f(x) = x^2$

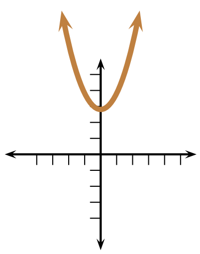
Figure 4.6: $y = x^2+3$

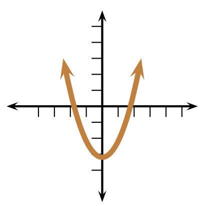
Figure 4.7: $y = x^2-3$

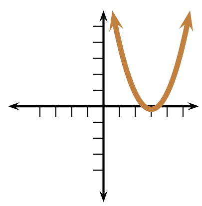
Figure 4.8: $y = (x-3)^2$

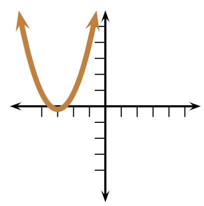
Figure 4.9: $y = (x+3)^2$

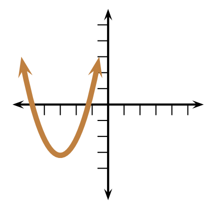
Figure 4.10: $y = (x+3)^2-3$

211 Example If $g(x) = x$ (figure 4.11), then figures , 4.12 and 4.13 shew vertical translations 3 units up and 3 units down, respectively. Notice than in this case $g(x+t) = x+t = g(x)+t$, so a vertical translation by t units has exactly the same graph as a horizontal translation t units.

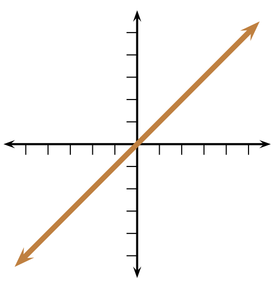
Figure 4.11: $y = g(x) = x$

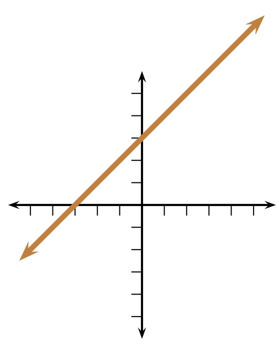
Figure 4.12: $y = g(x)+3 = x+3$

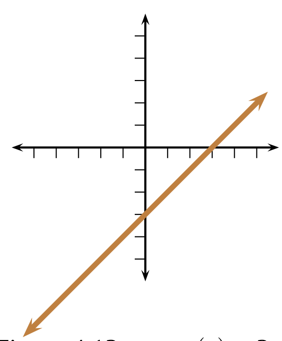
Figure 4.13: $y = g(x)-3 = x-3$

Homework

4.1.1 Problem Graph the following curves:

1. $y = |x-2|+3$
2. $y = (x-2)^2+3$
3. $y = \dfrac{1}{x-2}+3$
4. $y = \sqrt{4-x^2}+1$

4.1.2 Problem What is the equation of the curve $y = f(x) = x^3 - \dfrac{1}{x}$ after a successive translation one unit down and two units right?

4.1.3 Problem Suppose the curve $y = f(x)$ is translated a units vertically and b units horizontally, in this order. Would that have the same effect as translating the curve b units horizontally first, and then a units vertically?

4.2 Distortions

212 Theorem Let f be a function and let $V \neq 0$ and $H \neq 0$ be real numbers. If (x_0, y_0) is on the graph of f, then (x_0, Vy_0) is on the graph of g, where $g(x) = Vf(x)$, and if (x_1, y_1) is on the graph of f, then $\left(\dfrac{x_1}{H}, y_1\right)$ is on the graph of j, where $j(x) = f(Hx)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff Vy_0 = Vf(x_0) \iff Vy_0 = g(x_0) \iff (x_0, Vy_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f\left(\frac{x_1}{H} \cdot H\right) \iff y_1 = j\left(\frac{x_1}{H}\right) \iff \left(\frac{x_1}{H}, y_1\right) \in \Gamma_j.$$

❑

213 Definition Let $V > 0$, $H > 0$, and let f be a function. The curve $y = Vf(x)$ is called a *vertical distortion* of the curve $y = f(x)$. The graph of $y = Vf(x)$ is a *vertical dilatation* of the graph of $y = f(x)$ if $V > 1$ and a *vertical contraction* if $0 < V < 1$. The curve $y = f(Hx)$ is called a *horizontal distortion* of the curve $y = f(x)$ The graph of $y = f(Hx)$ is a *horizontal dilatation* of the graph of $y = f(x)$ if $0 < H < 1$ and a *horizontal contraction* if $H > 1$.

214 Example Consider the function

$$f : \begin{array}{ccc} [-4;4] & \to & [-6;6] \\ x & \mapsto & f(x) \end{array}$$

whose graph appears in figure 4.14.

If $a(x) = \dfrac{f(x)}{2}$ then

$$a : \begin{array}{ccc} [-4;4] & \to & [-3;3] \\ x & \mapsto & a(x) \end{array},$$

and its graph appears in figure 4.15.

If $b(x) = f(2x)$ then

$$b : \begin{array}{ccc} [-2;2] & \to & [-6;6] \\ x & \mapsto & b(x) \end{array},$$

and its graph appears in figure 4.16.

If $c(x) = \dfrac{f(2x)}{2}$ then

$$c : \begin{array}{ccc} [-2;2] & \to & [-3;3] \\ x & \mapsto & c(x) \end{array},$$

and its graph appears in figure 4.17.

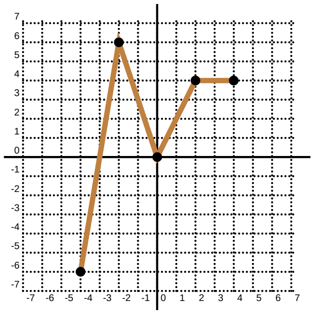

Figure 4.14: $y = f(x)$

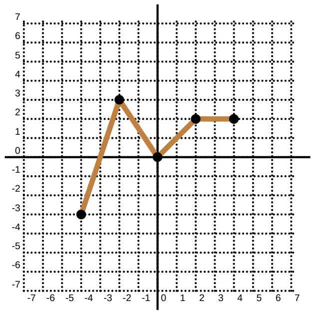

Figure 4.15: $y = \dfrac{f(x)}{2}$

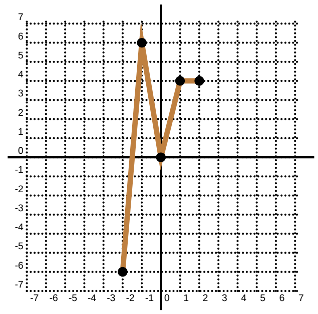

Figure 4.16: $y = f(2x)$

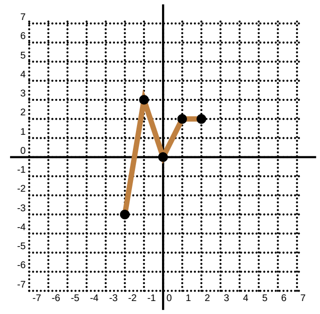

Figure 4.17: $y = \dfrac{f(2x)}{2}$

215 Example If $y = \sqrt{4-x^2}$, then $x^2+y^2 = 4$ gives the equation of a circle with centre at $(0,0)$ and radius 2 by virtue of 83. Hence

$$y = \sqrt{4-x^2}$$

is the upper semicircle of this circle. Figures 4.18 through 4.23 shew various transformations of this curve.

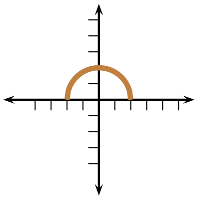

Figure 4.18: $y = \sqrt{4-x^2}$

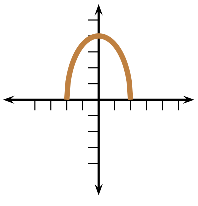

Figure 4.19: $y = 2\sqrt{4-x^2}$

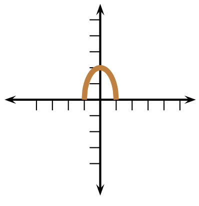

Figure 4.20: $y = \sqrt{4-4x^2}$

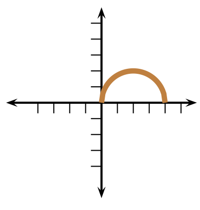

Figure 4.21: $y = \sqrt{-x^2+4x}$

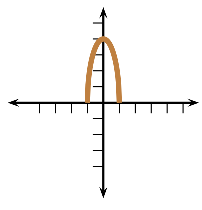

Figure 4.22: $y = 2\sqrt{4-4x^2}$

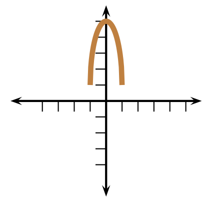

Figure 4.23: $y = 2\sqrt{4-4x^2}+1$

216 Example Draw the graph of the curve $y = 2x^2 - 4x + 1$.

Solution: ▶ *We complete squares.*

$$\begin{aligned} y = 2x^2 - 4x + 1 \quad &\iff \quad \frac{y}{2} = x^2 - 2x + \frac{1}{2} \\ &\iff \quad \frac{y}{2} + 1 = x^2 - 2x + 1 + \frac{1}{2} \\ &\iff \quad \frac{y}{2} + 1 = (x-1)^2 + \frac{1}{2} \\ &\iff \quad \frac{y}{2} = (x-1)^2 - \frac{1}{2} \\ &\iff \quad y = 2(x-1)^2 - 1, \end{aligned}$$

whence to obtain the graph of $y = 2x^2 - 4x + 1$ we (i) translate $y = x^2$ one unit right, (ii) dilate the above graph by factor of two, (iii) translate the above graph one unit down. This succession is seen in figures 4.24 through 4.26. ◀

217 Example The curve $y = x^2 + \frac{1}{x}$ experiences the following successive transformations: (i) a translation one unit up, (ii) a horizontal shrinkage by a factor of 2, (iii) a translation one unit left. Find its resulting equation.

Solution: ▶ *After a translation one unit up, the curve becomes*

$$y = f(x) + 1 = x^2 + \frac{1}{x} + 1 = a(x).$$

After a horizontal shrinkage by a factor of 2 the curve becomes

$$y = a(2x) = 4x^2 + \frac{1}{2x} + 1 = b(x).$$

After a translation one unit left the curve becomes

$$y = b(x+1) = 4(x+1)^2 + \frac{1}{2x+2} + 1.$$

The required equation is thus

$$y = 4(x+1)^2 + \frac{1}{2x+2} + 1 = 4x^2 + 8x + 5 + \frac{1}{2x+2}.$$

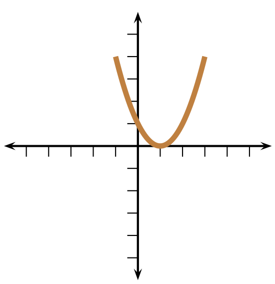

Figure 4.24: $y = (x-1)^2$

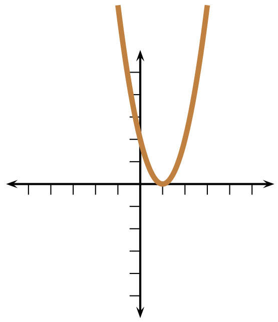

Figure 4.25: $y = 2(x-1)^2$

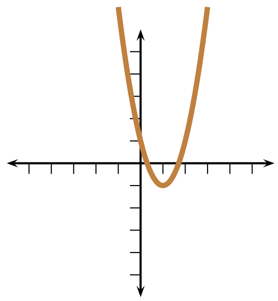

Figure 4.26: $y = 2(x-1)^2 - 1$

◀

Homework

4.2.1 Problem Draw the graphs of the following curves:

1. $y = \dfrac{x^2}{2}$
2. $y = \dfrac{x^2}{2} - 1$
3. $y = 2|x| + 1$
4. $y = \dfrac{2}{x}$
5. $y = x^2 + 4x + 5$
6. $y = 2x^2 + 8x$

4.2.2 Problem The curve $y = \dfrac{1}{x}$ experiences the following successive transformations: (i) a translation one unit left, (ii) a vertical dilatation by a factor of 2, (iii) a translation one unit down. Find its resulting equation and make a rough sketch of the resulting curve.

4.2.3 Problem For the functional curve given in figure 4.27, determine its domain and image and draw the following transformations, also determining their respective domains and images.

1. $y = 2f(x)$
2. $y = f(2x)$
3. $y = 2f(2x)$

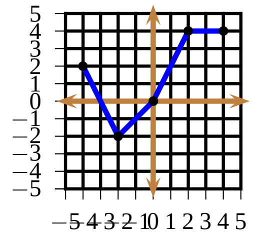

Figure 4.27: Problem 4.2.3.

4.3 Reflexions

218 Theorem Let f be a function If (x_0, y_0) is on the graph of f, then $(x_0, -y_0)$ is on the graph of g, where $g(x) = -f(x)$, and if (x_1, y_1) is on the graph of f, then $(-x_1, y_1)$ is on the graph of j, where $j(x) = f(-x)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff -y_0 = -f(x_0) \iff -y_0 = g(x_0) \iff (x_0, -y_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f(-(-x_1)) \iff y_1 = j(-x_1) \iff (-x_1, y_1) \in \Gamma_j.$$

❑

219 Definition Let f be a function. The curve $y = -f(x)$ is said to be the *reflexion of f about the x-axis* and the curve $y = f(-x)$ is said to be the *reflexion of f about the y-axis.*

220 Example Figure 4.28 shews the graph of the function

$$f : \begin{array}{ccc} [-4;4] & \to & [-2;4] \\ x & \mapsto & f(x) \end{array}.$$

Figure 4.29 shews the graph of its reflexion a about the x-axis,

$$a : \begin{array}{ccc} [-4;4] & \to & [-4;2] \\ x & \mapsto & a(x) \end{array}.$$

Figure 4.30 shews the graph of its reflexion b about the y-axis,

$$b : \begin{array}{ccc} [-4;4] & \to & [-2;4] \\ x & \mapsto & b(x) \end{array}.$$

Figure 4.31 shews the graph of its reflexion c about the x-axis and y-axis,

$$c : \begin{array}{ccc} [-4;4] & \rightarrow & [-4;2] \\ x & \mapsto & c(x) \end{array}.$$

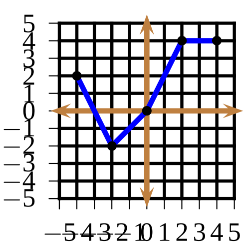

Figure 4.28: $y = f(x)$.

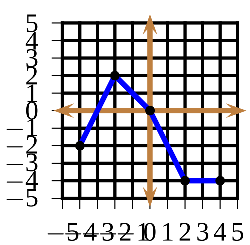

Figure 4.29: $y = -f(x)$.

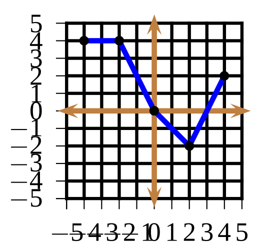

Figure 4.30: $y = f(-x)$.

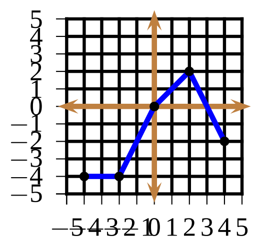

Figure 4.31: $y = -f(-x)$.

221 Example Figures 4.32 through 4.35 shew various reflexions about the axes for the function

$$d : \begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & (x-1)^2 \end{array}.$$

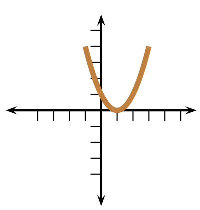

Figure 4.32: $y = d(x) = (x-1)^2$

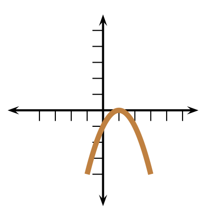

Figure 4.33: $y = -d(x) = -(x-1)^2$

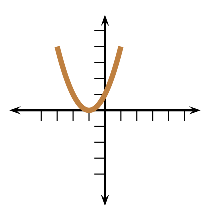

Figure 4.34: $y = d(-x) = (-x-1)^2$

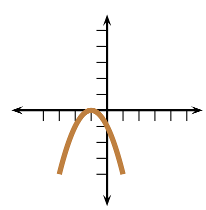

Figure 4.35: $y = -d(-x) = -(-x-1)^2$

222 Example Let $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ with

$$f(x) = x + \frac{2}{x} - 1.$$

The curve $y = f(x)$ experiences the following successive transformations:

1. A reflexion about the x-axis.
2. A translation 3 units left.
3. A reflexion about the y-axis.
4. A vertical dilatation by a factor of 2.

Find the equation of the resulting curve. Note also how the domain of the function is affected by these transformations.

Solution: ▶

1. *A reflexion about the x-axis gives the curve*

$$y = -f(x) = 1 - \frac{2}{x} - x = a(x),$$

   say, with $\mathbf{Dom}\,(a) = \mathbb{R} \setminus \{0\}$.

2. *A translation* 3 *units left gives the curve*

$$y = a(x+3) = 1 - \frac{2}{x+3} - (x+3) = -2 - \frac{2}{x+3} - x = b(x),$$

   say, with $\mathbf{Dom}\,(b) = \mathbb{R} \setminus \{-3\}$.

3. *A reflexion about the y-axis gives the curve*

$$y = b(-x) = -2 - \frac{2}{-x+3} + x = c(x),$$

   say, with $\mathbf{Dom}\,(c) = \mathbb{R} \setminus \{3\}$.

4. *A vertical dilatation by a factor of* 2 *gives the curve*

$$y = 2c(x) = -4 + \frac{4}{x-3} + 2x = d(x),$$

   say, with $\mathbf{Dom}\,(d) = \mathbb{R} \setminus \{3\}$. *Notice that the resulting curve is*

$$y = d(x) = 2c(x) = 2b(-x) = 2a(-x+3) = -2f(-x+3).$$

◀

Homework

4.3.1 Problem Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = 2 - |x|.$$

The curve $y = f(x)$ experiences the following successive transformations:

1. A reflexion about the x-axis.
2. A translation 3 units up.
3. A horizontal stretch by a factor of $\frac{3}{4}$.

Find the equation of the resulting curve.

4.3.2 Problem The graphs of the following curves suffer the following successive, rigid transformations:

1. a vertical translation of 2 units down,
2. a reflexion about the y-axis, and finally,
3. a horizontal translation of 1 unit to the left.

Find the resulting equations after all the transformations have been exerted.

1. $y = x(1-x)$
2. $y = 2x - 3$
3. $y = |x+2| + 1$

4.3.3 Problem For the functional curve $y = f(x)$ in figure 4.36, draw $y = f(x+1)$, $y = f(1-x)$ and $y = -f(1-x)$.

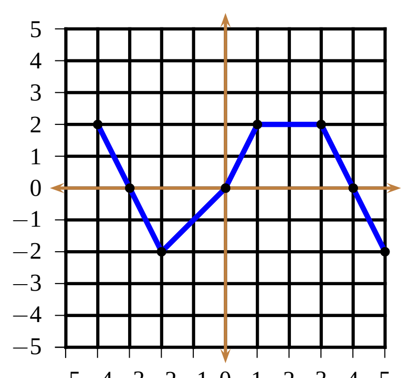

Figure 4.36: Problem 4.3.3.

4.4 Symmetry

223 Definition A function f is *even* if for all x it is verified that $f(x) = f(-x)$, that is, if the portion of the graph for $x < 0$ is a mirror reflexion of the part of the graph for $x > 0$. This means that the graph of f is symmetric about the y-axis. A function g is *odd* if for all x it is verified that $g(-x) = -g(x)$, in other words, g is odd if it is symmetric about the origin. This implies that the portion of the graph appearing in quadrant I is a 180° rotation of the portion of the graph appearing in quadrant III, and the portion of the graph appearing in quadrant II is a 180° rotation of the portion of the graph appearing in quadrant IV.

224 Example The curve in figure 4.37 is even. The curve in figure 4.38 is odd.

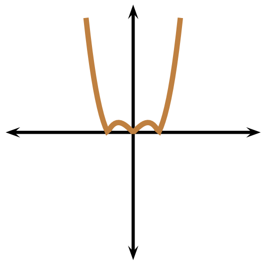

Figure 4.37: Example 224. The graph of an even function.

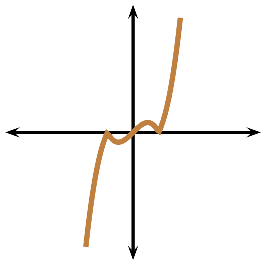

Figure 4.38: Example 224. The graph of an odd function.

225 Theorem Let $\varepsilon_1, \varepsilon_2$ be even functions, and let ω_1, ω_2 be odd functions, all sharing the same common domain. Then

1. $\varepsilon_1 \pm \varepsilon_2$ is an even function.
2. $\omega_1 \pm \omega_2$ is an odd function.
3. $\varepsilon_1 \cdot \varepsilon_2$ is an even function.
4. $\omega_1 \cdot \omega_2$ is an even function.
5. $\varepsilon_1 \cdot \omega_1$ is an odd function.

Proof: *We have*

1. $(\varepsilon_1 \pm \varepsilon_2)(-x) = \varepsilon_1(-x) \pm \varepsilon_2(-x) = \varepsilon_1(x) \pm \varepsilon_2(x)$.
2. $(\omega_1 \pm \omega_2)(-x) = \omega_1(-x) \pm \omega_2(-x) = -\omega_1(x) \mp \omega_2(x) = -(\omega_1 \pm \omega_2)(x)$
3. $(\varepsilon_1\varepsilon_2)(-x) = \varepsilon_1(-x)\varepsilon_2(-x) = \varepsilon_1(x)\varepsilon_2(x)$
4. $(\omega_1\omega_2)(-x) = \omega_1(-x)\omega_2(-x) = (-\omega_1(x))(-\omega_2(x)) = \omega_1(x)\omega_2(x))$
5. $(\varepsilon_1\omega_1)(-x) = \varepsilon_1(-x)\omega_1(-x) = -\varepsilon_1(x)\omega_1(x)$

❑

226 Corollary Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with real coefficients. Then the function

$$p : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & p(x) \end{array}$$

is an even function if and only if each of its terms has even degree.

Proof: *Assume p is even. Then $p(x) = p(-x)$ and so*

$$\begin{aligned} p(x) &= \frac{p(x)+p(-x)}{2} \\ &= \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n}{2} \\ &\quad + \frac{a_0 - a_1x + a_2x^2 - a_3x^3 + \cdots + (-1)^{n-1}a_{n-1}x^{n-1} + (-1)^n a_nx^n}{2} \\ &= a_0 + a_2x^2 + a_4x^4 + \cdots + \end{aligned}$$

and so the polynomial has only terms of even degree. The converse of this statement is trivial. ❑

227 Example Prove that in the product

$$(1-x+x^2-x^3+\cdots-x^{99}+x^{100})(1+x+x^2+x^3+\cdots+x^{99}+x^{100})$$

after multiplying and collecting terms, there does not appear a term in x of odd degree.

Solution: ► *Let* $f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & f(x) \end{array}$ *with*

$$f(x) = (1-x+x^2-x^3+\cdots-x^{99}+x^{100})(1+x+x^2+x^3+\cdots+x^{99}+x^{100})$$

Then

$$f(-x) = (1+x+x^2+x^3+\cdots+x^{99}+x^{100})(1-x+x^2-x^3+\cdots-x^{99}+x^{100}) = f(x),$$

which means that f is an even function. Since f is a polynomial, this means that f does not have a term of odd degree. ◄

Analogous to Corollary 226, we may establish the following.

228 Corollary Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with real coefficients. Then the function

$$p: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & p(x) \end{array}$$

is an odd function if and only if each of its terms has odd degree.

229 Theorem Let $f: \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then f can be written as the sum of an even function and an odd function.

Proof: *Given $x \in \mathbb{R}$, put $E(x) = f(x) + f(-x)$, and $O(x) = f(x) - f(-x)$. We claim that E is an even function and that O is an odd function. First notice that*

$$E(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = E(x),$$

which proves that E is even. Also,

$$O(-x) = f(-x) - f(-(-x)) = -(f(x) - f(-x)) = -O(x),$$

which proves that O is an odd function. Clearly

$$f(x) = \frac{1}{2}E(x) + \frac{1}{2}O(x),$$

which proves the theorem. ❑

230 Example Investigate which of the following functions are even, odd, or neither.

1. $a: \mathbb{R} \to \mathbb{R}$, $a(x) = \dfrac{x^3}{x^2+1}$.
2. $b: \mathbb{R} \to \mathbb{R}$, $b(x) = \dfrac{|x|}{x^2+1}$.
3. $c: \mathbb{R} \to \mathbb{R}$, $c(x) = |x| + 2$.
4. $d: \mathbb{R} \to \mathbb{R}$, $d(x) = |x+2|$.

5. $f : [-4;5] \to \mathbb{R}$, $f(x) = |x| + 2$.

Solution: ►

1.
$$a(-x) = \frac{(-x)^3}{(-x)^2+1} = -\frac{x^3}{x^2+1} = -a(x),$$
whence a is odd, since its domain is also symmetric.

2.
$$b(-x) = \frac{|-x|}{(-x)^2+1} = \frac{|x|}{x^2+1} = b(x),$$
whence b is even, since its domain is also symmetric.

3.
$$c(-x) = |-x| + 2 = |x| + 2 = c(x),$$
whence c is even, since its domain is also symmetric.

4. $d(-1) = |-1+2| = 1$, but $d(1) = 3$. This function is neither even nor odd.

5. The domain of f is not symmetric, so f is neither even nor odd.

◄

Homework

4.4.1 Problem Complete the following fragment of graph so that the completion depicts (i) an even function, (ii) an odd function.

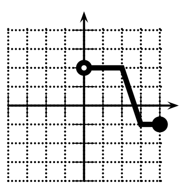

Figure 4.39: Problem 4.4.1.

4.4.2 Problem Let $f : \mathbb{R} \to \mathbb{R}$ be an even function and let $g : \mathbb{R} \to \mathbb{R}$ be an odd function. If $f(-2) = 3$, $f(3) = 2$ and $g(-2) = 2$, $g(3) = 4$, find
$$(f+g)(2), \qquad (g \circ f)(2).$$

4.4.3 Problem Let f be an odd function and assume that f is defined at $x = 0$. Prove that $f(0) = 0$.

4.4.4 Problem Can a function be simultaneously even and odd? What would the graph of such a function look like?

4.4.5 Problem Let $A \times B \subseteq \mathbb{R}^2$ and suppose that $f : A \to B$ is invertible and even. Determine the sets A and B.

4.5 Transformations Involving Absolute Values

231 Theorem Let f be a function. Then both $x \mapsto f(|x|)$ and $x \mapsto f(-|x|)$ are even functions.

Proof: *Put $a(x) = f(|x|)$. Then $a(-x) = f(|-x|) = f(|x|) = a(x)$, whence $x \mapsto a(x)$ is even. Similarly, if $b(x) = f(-|x|)$, then $b(-x) = f(-|-x|) = f(-|x|) = b(x)$ proving that $x \mapsto b(x)$ is even.* ❑

Notice that $f(x) = f(|x|)$ for $x > 0$. Since $x \mapsto f(|x|)$ is even, the graph of $x \mapsto f(|x|)$ is thus obtained by erasing the portion of the graph of $x \mapsto f(x)$ for $x < 0$ and reflecting the part for $x > 0$. Similarly, since $f(x) = f(-|x|)$ for $x < 0$, the graph of $x \mapsto f(-|x|)$ is obtained by erasing the portion of the graph of $x \mapsto f(x)$ for $x > 0$ and reflecting the part for $x < 0$.

232 Theorem Let f be a function If (x_0, y_0) is on the graph of f, then $(x_0, |y_0|)$ is on the graph of g, where $g(x) = |f(x)|$.

Proof: *Let Γ_f, Γ_g denote the graphs of f, g, respectively.*
$$(x_0, y_0) \in \Gamma_f \implies y_0 = f(x_0) \implies |y_0| = |f(x_0)| \implies |y_0| = g(x_0) \implies (x_0, |y_0|) \in \Gamma_g.$$
❑

233 Example The graph of $y = f(x)$ is given in figure 4.40. The transformation $y = |f(x)|$ is given in figure 4.41. The transformation $y = f(|x|)$ is given in figure 4.42. The transformation $y = f(-|x|)$ is given in figure 4.43. The transformation $y = |f(|x|)|$ is given in figure 4.44.

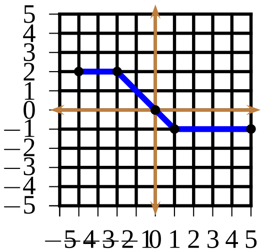

Figure 4.40: $y = f(x)$.

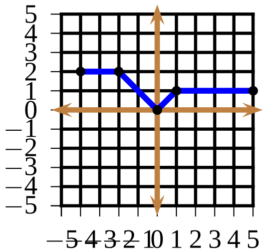

Figure 4.41: $y = |f(x)|$.

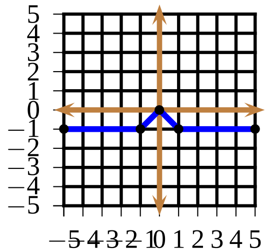

Figure 4.42: $y = f(|x|)$.

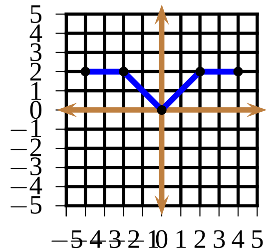

Figure 4.43: $y = f(-|x|)$.

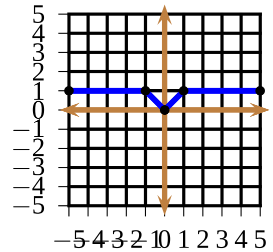

Figure 4.44: $y = |f(|x|)|$.

234 Example Figures 4.45 through 4.48 exhibit various transformations of $f : x \mapsto (x-1)^2 - 3$.

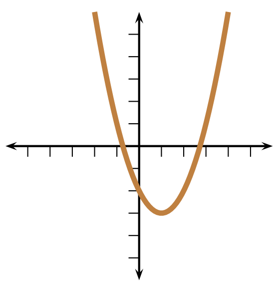

Figure 4.45: $y = f(x) = (x-1)^2 - 3$

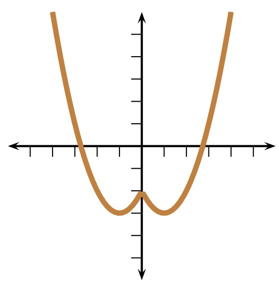

Figure 4.46: $y = f(|x|) = (|x|-1)^2 - 3$

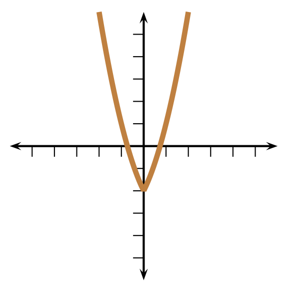

Figure 4.47: $y = f(-|x|) = (-|x|-1)^2 - 3$

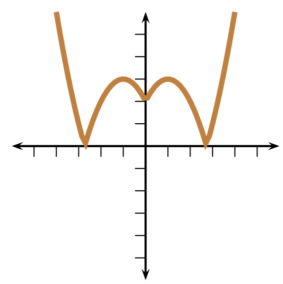

Figure 4.48: $y = |f(|x|)| = |(|x|-1)^2 - 3|$

Homework

4.5.1 Problem Use the graph of f in figure 4.49 in order to draw

1. $y = 2f(x)$
2. $y = f(2x)$
3. $y = f(-x)$
4. $y = -f(x)$
5. $y = -f(-x)$
6. $y = f(|x|)$
7. $y = |f(x)|$
8. $y = f(-|x|)$

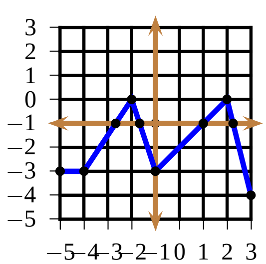

Figure 4.49: $y = f(x)$

4.5.2 Problem Draw the graph of the curve $y = \sqrt{|x|}$.

4.5.3 Problem Draw the curves $y = x^2 - 1$ and $y = |x^2 - 1|$ in succession.

4.5.4 Problem Draw the graphs of the curves

$$y = \sqrt{-x^2 + 2|x| + 3}, \qquad y = \sqrt{-x^2 - 2|x| + 3}.$$

4.5.5 Problem Draw the following graphs in succession.

1. $y = (x-1)^2 - 2$
2. $y = |(x-1)^2 - 2|$
3. $y = (|x|-1)^2 - 2$
4. $y = (1+|x|)^2 - 2$

4.5.6 Problem Draw the graph of $f : \mathbb{R} \to \mathbb{R}$, with assignment rule $f(x) = x|x|$.

4.5.7 Problem Draw the following curves in succession:

1. $y = x^2$
2. $y = (x-1)^2$
3. $y = (|x|-1)^2$

4.5.8 Problem Draw the following curves in succession:

1. $y = x^2$
2. $y = x^2 - 1$
3. $y = |x^2 - 1|$

4.5.9 Problem Draw the following curves in succession:

1. $y = x^2 + 2x + 3$
2. $y = x^2 + 2|x| + 3$
3. $y = |x^2 + 2x + 3|$
4. $y = |x^2 + 2|x| + 3|$

4.5.10 Problem Draw the following curves in succession:

1. $y = 1 - x$
2. $y = |1 - x|$
3. $y = 1 - |1 - x|$
4. $y = |1 - |1 - x||$
5. $y = 1 - |1 - |1 - x||$
6. $y = |1 - |1 - |1 - x|||$
7. $y = 1 - |1 - |1 - |1 - x|||$
8. $y = |1 - |1 - |1 - |1 - x||||$

4.5.11 Problem Put $f_1(x) = x$; $f_2(x) = |1 - f_1(x)|$; $f_3(x) = |1 - f_2(x)|$; $\ldots$ $f_n(x) = |1 - f_{n-1}(x)|$. Prove that the solutions of the equation $f_n(x) = 0$ are $\{\pm 1, \pm 3, \ldots, \pm(n-3), (n-1)\}$ if n is even and $\{0, \pm 2, \ldots, \pm(n-3), (n-1)\}$ if n is odd.

4.5.12 Problem Given in figures 4.50 and 4.51 are the graphs of two curves, $y = f(x)$ and $y = f(ax)$ for some real constant $a < 0$.

1. Determine the value of the constant a.
2. Determine the value of C.

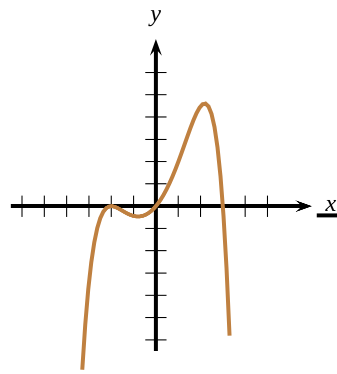

Figure 4.50: Problem 4.5.12. $y = f(x)$

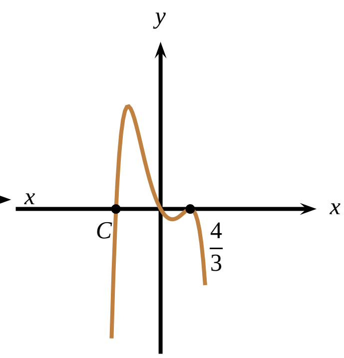

Figure 4.51: Problem 4.5.12. $y = f(ax)$

4.6 Behaviour of the Graphs of Functions

So far we have limited our study of functions to those families of functions whose graphs are known to us: lines, parabolas, hyperbolas, or semicircles. Through some arguments involving symmetry we have been able to extend this collection to compositions of the above listed functions with the absolute value function. We would now like to increase our repertoire of functions that we can graph. For that we need the machinery of Calculus, which will be studied in subsequent courses. We will content ourselves with *informally* introducing various terms useful when describing curves and with proving that these properties hold for some simple curves.

4.6.1 Continuity

235 Definition We write $x \to a+$ to indicate the fact that x is progressively getting closer and closer to a through values greater (to the right) of a. Similarly, we write $x \to a-$ to indicate the fact that x is progressively getting closer and closer to a through values smaller (to the left) of a. Finally, we write $x \to a$ to indicate the fact that x is progressively getting closer and closer to a through values left and right of a.

236 Definition Given a function f, we write $f(a+)$ for the value that $f(x)$ approaches as $x \to a+$. In other words, we consider the values of a dextral neighbourhood of a, progressively decrease the length of this neighbourhood, and see which value f approaches in this neighbourhood. Similarly, we write $f(a-)$ for the value that $f(x)$ approaches as $x \to a-$. In other words, we consider the values of a sinistral neighbourhood of a, progressively decrease the length of this neighbourhood, and see which value f approaches in this neighbourhood.

237 Example Let $f : [-4;4] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x^2+1 & \text{if } -4 \leq x < -2 \\ 2 & \text{if } x = -2 \\ 2+2x & \text{if } -2 < x < +2 \\ 6 & \text{if } +2 \leq x \leq 4 \end{cases}$$

Determine

1. $f(-2-)$
2. $f(-2)$
3. $f(-2+)$
4. $f(+2-)$
5. $f(+2)$
6. $f(+2+)$

Solution: ►

1. *To find* $f(-2-)$ *we look at the definition of* f *just to the left of* -2*. Thus* $f(-2-) = (-2)^2 + 1 = 5$.
2. $f(-2) = 2$.
3. *To find* $f(-2+)$ *we look at the definition of* f *just to the right of* -2*. Thus* $f(-2+) = 2+2(-2) = -2$.
4. *To find* $f(+2-)$ *we look at the definition of* f *just to the left of* $+2$*. Thus* $f(+2-) = 2+2(2) = 6$.
5. $f(+2) = 6$.
6. *To find* $f(+2+)$ *we look at the definition of* f *just to the right of* $+2$*. Thus* $f(+2+) = 6$.

◄

Let us consider the following situation. Let f be a function and $a \in \mathbb{R}$. Assume that f is defined in a neighbourhood of a, but not precisely at $x = a$. Which value can we reasonably assign to $f(a)$? Consider the situations depicted in figures 4.52 through 4.54. In figure 4.52 it seems reasonably to assign $a(0) = 0$. What value can we reasonably assign in figure 4.53? $b(0) = \dfrac{-1+1}{2} = 0$? In figure 4.54, what value would it be reasonable to assign? $c(0) = 0$?, $c(0) = +\infty$?, $c(0) = -\infty$? The situations presented here are typical, but not necessarily exhaustive.

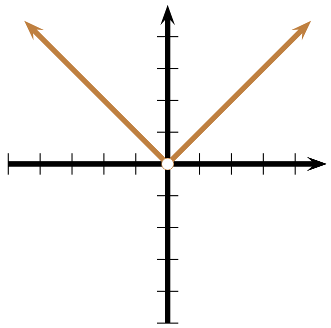

Figure 4.52: $a : x \mapsto |x|, x \neq 0$.

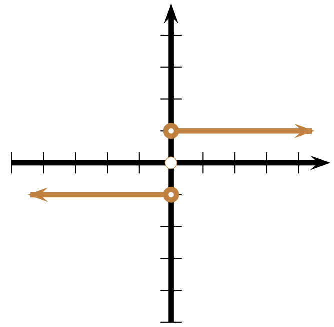

Figure 4.53: $b : x \mapsto \dfrac{x}{|x|}, x \neq 0$.

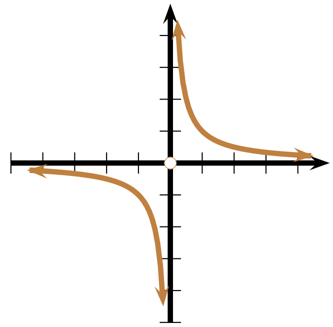

Figure 4.54: $c : x \mapsto \dfrac{1}{x}, x \neq 0$.

238 Definition A function f is said to be *left continuous* at the point $x = a$ if $f(a-) = f(a)$. A function f is said to be *right continuous* at the point $x = a$ if $f(a) = f(a+)$. A function f is said to be *continuous* at the point $x = a$ if $f(a-) = f(a) = f(a+)$. It is continuous on the interval I if it is continuous on every point of I.

Heuristically speaking, a continuous function is one whose graph has no "breaks."

239 Example Given that

$$f(x) = \begin{cases} 6+x & \text{if } x \in]-\infty;-2] \\ 3x^2+xa & \text{if } x \in]-2;+\infty[\end{cases}$$

is continuous, find a.

Solution: ► *Since* $f(-2-) = f(-2) = 6-2 = 4$ *and* $f(-2+) = 3(-2)^2 - 2a = 12-2a$ *we need*

$$f(-2-) = f(-2+) \implies 4 = 12-2a \implies a = 4.$$

◄

4.6.2 Monotonicity

240 Definition A function f is said to be *increasing* (respectively, *strictly increasing*) if $a < b \implies f(a) \leq f(b)$ (respectively, $a < b \implies f(a) < f(b)$). A function g is said to be *decreasing* (respectively, *strictly decreasing*) if $a < b \implies g(a) \leq g(b)$ (respectively, $a < b \implies g(a) < g(b)$). A function is *monotonic* if it is either (strictly) increasing or decreasing. By the *intervals of monotonicity of a function* we mean the intervals where the function might be (strictly) increasing or decreasing.

If the function f is (strictly) increasing, its opposite $-f$ is (strictly) decreasing, and viceversa.

The following theorem is immediate.

241 Theorem A function f is (strictly) increasing if for all $a < b$ for which it is defined

$$\frac{f(b)-f(a)}{b-a} \geq 0 \qquad \text{(respectively, } \frac{f(b)-f(a)}{b-a} > 0).$$

Similarly, a function g is (strictly) decreasing if for all $a < b$ for which it is defined

$$\frac{g(b)-g(a)}{b-a} \leq 0 \qquad \text{(respectively, } \frac{g(b)-g(a)}{b-a} < 0).$$

4.6.3 Extrema

242 Definition If there is a point a for which $f(x) \leq f(M)$ for all x in a neighbourhood centred at $x = M$ then we say that f has a *local maximum* at $x = M$. Similarly, if there is a point m for which $f(x) \geq f(m)$ for all x in a neighbourhood centred at $x = m$ then we say that f has a *local minimum* at $x = m$. The maxima and the minima of a function are called its *extrema*.

Consider now a continuous function in a closed interval $[a;b]$. Unless it is a horizontal line there, its graph goes up and down in $[a;b]$. It cannot go up forever, since otherwise it would be unbounded and hence not continuous. Similarly, it cannot go down forever. Thus there exist α, β in $[a;b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$, that is, f reaches maxima and minima in $[a;b]$.

4.6.4 Convexity

We now investigate define the "bending" of the graph of a function.

243 Definition A function $f : A \to B$ is *convex* in A if $\forall (a,b,\lambda) \in A^2 \times [0;1]$,

$$f(\lambda a + (1-\lambda)b) \leq f(a)\lambda + (1-\lambda)f(b).$$

Similarly, a function $g : A \to B$ is *concave* in A if $\forall (a,b,\lambda) \in A^2 \times [0;1]$,

$$g(\lambda a + (1-\lambda)b) \geq g(a)\lambda + (1-\lambda)g(b).$$

By the *intervals of convexity (concavity) of a function* we mean the intervals where the function is convex (concave). An *inflexion point* is a point where a graph changes convexity.

By Lemma 15, $\lambda a + (1-\lambda)b$ lies in the interval $[a;b]$ for $0 \leq \lambda \leq 1$. Hence, geometrically speaking, a convex function is one such that if two distinct points on its graph are taken and the straight line joining these two points drawn, then the midpoint of that straight line is above the graph. In other words, the graph of the function bends upwards. Notice that if f is convex, then its opposite $-f$ is concave.

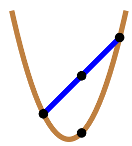

Figure 4.55: A convex curve

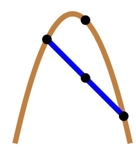

Figure 4.56: A concave curve.

Homework

4.6.1 Problem Given that

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

is continuous, find a.

4.6.2 Problem Give an example of a function which is discontinuous on the set $\{-1, 0, 1\}$ but continuous everywhere else.

4.6.3 Problem Given that

$$f(x) = \begin{cases} x^2-1 & \text{if } x \leq 1 \\ 2x+3a & \text{if } x > 1 \end{cases}$$

is continuous, find a.

4.6.4 Problem Let n be a strictly positive integer. Given that

$$f(x) = \begin{cases} \dfrac{x^n-1}{x-1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

is continuous, find a.

4.6.5 Problem Give an example of a function discontinuous at the points $\pm\sqrt[3]{1}, \pm\sqrt[3]{2}, \pm\sqrt[3]{3}, \pm\sqrt[3]{4}, \pm\sqrt[3]{5}, \ldots$.

4.7 The functions $x \mapsto \lfloor x \rfloor$, $x \mapsto \lceil x \rceil$, $x \mapsto \{x\}$

244 Definition The *floor* $\lfloor x \rfloor$ of a real number x is the unique integer defined by the inequality

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

In other words, $\lfloor x \rfloor$ is x if x is an integer, or the integer just to the left, if x is not an integer. For example

$$\lfloor 3 \rfloor = 3, \qquad \lfloor 3.9 \rfloor = 3, \qquad \lfloor -\pi \rfloor = -4.$$

If $n \in \mathbb{Z}$ and if

$$n \leq x < n+1,$$

then $\lfloor x \rfloor = n$. This means that the function $x \mapsto \lfloor x \rfloor$ is constant between two consecutive integers. For example, between 0 and 1 it will have output 0; between 1 and 2, it will have output 1, etc., always taking the smaller of the two consecutive integers. Its graph has the staircase shape found in figure 4.57.

245 Definition The *ceiling* $\lceil x \rceil$ of a real number x is the unique integer defined by the inequality

$$\lceil x \rceil - 1 < x \leq \lceil x \rceil.$$

In other words, $\lceil\!\lceil x \rceil\!\rceil$ is x if x is an integer, or the integer just to the right, if x is not an integer. For example

$$\lceil\!\lceil 3 \rceil\!\rceil = 3, \qquad \lceil\!\lceil 3.9 \rceil\!\rceil = 4, \qquad \lfloor\!\lfloor -\pi \rfloor\!\rfloor = -3.$$

If $n \in \mathbb{Z}$ and if

$$n < x \leq n+1,$$

then $\lceil\!\lceil x \rceil\!\rceil = n+1$. This means that the function $x \mapsto \lceil\!\lceil x \rceil\!\rceil$ is constant between two consecutive integers. For example, between 0 and 1 it will have output 1; between 1 and 2, it will have output 2, etc., always taking the larger of the two consecutive integers. Its graph has the staircase shape found in figure 4.58.

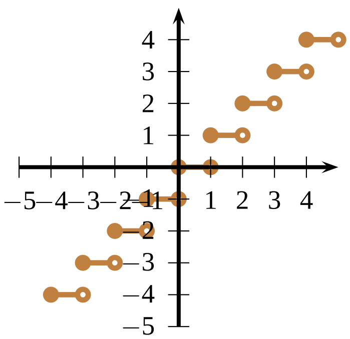

Figure 4.57: $x \mapsto \lfloor\!\lfloor x \rfloor\!\rfloor$.

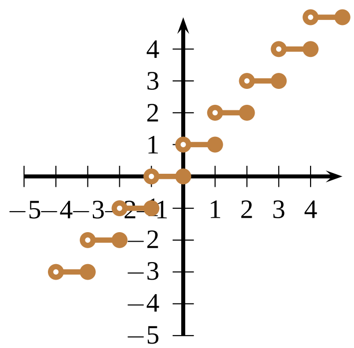

Figure 4.58: $x \mapsto \lceil\!\lceil x \rceil\!\rceil$.

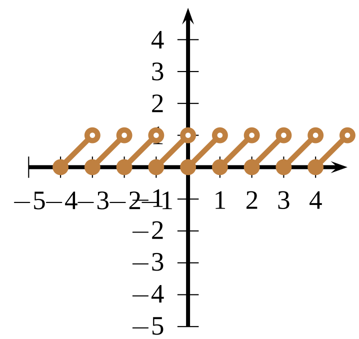

Figure 4.59: $x \mapsto x - \lfloor\!\lfloor x \rfloor\!\rfloor$.

246 Definition A function f is said to be *periodic of period P* if there a real number $P > 0$ such that

$$x \in \mathbf{Dom}(f) \implies (x+P) \in \mathbf{Dom}(f), \quad f(x+P) = f(x).$$

That is, if f is periodic of period P then once f is defined on an interval of period P, then it will be defined for all other values of its domain.

The discussion below will make use of the following lemma.

247 Lemma Let $x \in \mathbb{R}$ and $z \in \mathbb{Z}$. Then

$$\lfloor\!\lfloor x+z \rfloor\!\rfloor = \lfloor\!\lfloor x \rfloor\!\rfloor + z.$$

Proof: *Recall that $\lfloor\!\lfloor x \rfloor\!\rfloor$ is the unique integer with the property*

$$\lfloor\!\lfloor x \rfloor\!\rfloor \leq x < \lfloor\!\lfloor x \rfloor\!\rfloor + 1.$$

In turn, this means that $\lfloor\!\lfloor x+z \rfloor\!\rfloor - z$ also satisfies this inequality.

By definition,

$$\lfloor\!\lfloor x+z \rfloor\!\rfloor \leq x+z < \lfloor\!\lfloor x+z \rfloor\!\rfloor + 1,$$

and so we have,

$$\lfloor\!\lfloor x+z \rfloor\!\rfloor - z \leq x < \lfloor\!\lfloor x+z \rfloor\!\rfloor - z + 1,$$

from where $\lfloor\!\lfloor x+z \rfloor\!\rfloor - z$ satisfies the desired inequality and we conclude that e $\lfloor\!\lfloor x+z \rfloor\!\rfloor - z = \lfloor\!\lfloor x \rfloor\!\rfloor$, demonstrating theorem. ❑

248 Example Put $\{x\} = x - \lfloor\!\lfloor x \rfloor\!\rfloor$. Consider the function $f : \mathbb{R} \to [0;1[$, $f(x) = \{x\}$, the decimal part *decimal part of x*. We have

$$\lfloor\!\lfloor x \rfloor\!\rfloor \leq x < \lfloor\!\lfloor x \rfloor\!\rfloor + 1 \implies 0 \leq x - \lfloor\!\lfloor x \rfloor\!\rfloor < 1.$$

Also, by virtue of lemma 247,

$$f(x+1) = \{x+1\} = (x+1) - \lfloor\!\lfloor x+1 \rfloor\!\rfloor = (x+1) - (\lfloor\!\lfloor x \rfloor\!\rfloor + 1) = x - \lfloor\!\lfloor x \rfloor\!\rfloor = \{x\} = f(x),$$

which means that f is periodic of period 1. Now,

$$x \in [0;1[\implies \{x\} = x,$$

from where we gather that between 0 and 1, f behaves like the identity function. The graph of $x \mapsto \{x\}$ appears in figure 4.59 .

Homework

4.7.1 Problem Give an example of a function r discontinuous at the reciprocal of every non-zero integer.

4.7.2 Problem Give an example of a function discontinuous at the odd integers.

4.7.3 Problem Give an example of a function discontinuous at the square of every integer.

4.7.4 Problem Let $||x|| = \min_{n\in\mathbb{Z}} |x-n|$. Prove that $x \mapsto ||x||$ is periodic and find its period. Also, graph this function. Notice that this function measures the distance of a real number to its nearest integer.

4.7.5 Problem Investigate the graph of $x \mapsto \llbracket 2x \rrbracket$.

4.7.6 Problem Is it true that for all real numbers x we have $\{x^2\} = \{x\}^2$?

4.7.7 Problem Demonstrate that the function $f : \mathbb{R} \to \{-1, 1\}$ given by $f(x) = (-1)^{\llbracket x \rrbracket}$ is periodic of period 2 and draw its graph.

4.7.8 Problem Discuss the graph of $x \mapsto \dfrac{1}{\lceil\!\lceil x \rceil\!\rceil - \llbracket x \rrbracket}$.

4.7.9 Problem Find the points of discontinuity of the function $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto \llbracket x \rrbracket + \sqrt{x - \llbracket x \rrbracket}$.

4.7.10 Problem Find the points of discontinuity of the function

$$f : \begin{array}{ccc} x & \to & \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ \mathbb{R} & \mapsto & \mathbb{R} \end{array}.$$

4.7.11 Problem Find the points of discontinuity of the function

$$f : \begin{array}{ccc} x & \to & \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ \mathbb{R} & \mapsto & \mathbb{R} \end{array}.$$

4.7.12 Problem Find the points of discontinuity of the function

$$f : \begin{array}{ccc} x & \to & \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ \mathbb{R} & \mapsto & \mathbb{R} \end{array}.$$

4.7.13 Problem Prove that $f : \mathbb{R} \to \mathbb{R}$, $f(t+1) = \dfrac{1}{2} + \sqrt{f(t) - (f(t))^2}$ has period 2.

5 Polynomial Functions

249 Definition A *polynomial* $p(x)$ of degree $n \in \mathbb{N}$ is an expression of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0, \qquad a_k \in \mathbb{R},$$

where the a_k are constants. If the a_k are all integers then we say that p has integer coefficients, and we write $p(x) \in \mathbb{Z}[x]$; if the a_k are real numbers then we say that p has real coefficients and we write $p(x) \in \mathbb{R}[x]$; etc. The degree n of the polynomial p is denoted by $\deg p$. The coefficient a_n is called the *leading coefficient* of $p(x)$. A *root* of p is a solution to the equation $p(x) = 0$.

In this chapter we learn how to graph polynomials all whose roots are real numbers.

250 Example Here are a few examples of polynomials.

- $a(x) = 2x + 1 \in \mathbb{Z}[x]$, is a polynomial of degree 1, and leading coefficient 2. It has $x = -\dfrac{1}{2}$ as its only root. A polynomial of degree 1 is also known as an *affine function*.
- $b(x) = \pi x^2 + x - \sqrt{3} \in \mathbb{R}[x]$, is a polynomial of degree 2 and leading coefficient π. By the quadratic formula b has the two roots
$$x = \frac{-1 + \sqrt{1 + 4\pi\sqrt{3}}}{2\pi} \qquad \text{and} \qquad x = \frac{-1 - \sqrt{1 + 4\pi\sqrt{3}}}{2\pi}.$$
A polynomial of degree 2 is also called a *quadratic polynomial* or *quadratic function*.
- $C(x) = 1 \cdot x^0 := 1$, is a constant polynomial, of degree 0. It has no roots, since it is never zero.

251 Theorem The degree of the product of two polynomials is the sum of their degrees. In symbols, if p, q are polynomials, $\deg pq = \deg p + \deg q$.

Proof: *If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, with $a_n \neq 0$ and $b_m \neq 0$ then upon multiplication,*

$$p(x)q(x) = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0) = a_n b_m x^{m+n} + \cdots +,$$

with non-vanishing leading coefficient $a_n b_m$. ❑

252 Example The polynomial $p(x) = (1 + 2x + 3x^3)^4(1 - 2x^2)^5$ has leading coefficient $3^4(-2)^5 = -2592$ and degree $3 \cdot 4 + 2 \cdot 5 = 22$.

253 Example What is the degree of the polynomial identically equal to 0? Put $p(x) \equiv 0$ and, say, $q(x) = x + 1$. Then by Theorem 251 we must have $\deg pq = \deg p + \deg q = \deg p + 1$. But pq is identically 0, and hence $\deg pq = \deg p$. But if $\deg p$ were finite then

$$\deg p = \deg pq = \deg p + 1 \implies 0 = 1,$$

nonsense. Thus the 0-polynomial does not have any finite degree. We attach to it, by convention, degree $-\infty$.

5.1 Power Functions

254 Definition A *power function* is a function whose formula is of the form $x \mapsto x^\alpha$, where $\alpha \in \mathbb{R}$. In this chapter we will only study the case when α is a positive integer.

If n is a positive integer, we are interested in how to graph $x \mapsto x^n$. We have already encountered a few instances of power functions. For $n = 0$, the function $x \mapsto 1$ is a constant function, whose graph is the straight line $y = 1$ parallel to the x-axis. For $n = 1$, the function $x \mapsto x$ is the identity function, whose graph is the straight line $y = x$, which bisects the first and third quadrant. These graphs were not obtained by fiat, we demonstrated that the graphs are indeed straight lines in Theorem 93. Also, for $n = 2$, we have the square function $x \mapsto x^2$ whose graph is the parabola $y = x^2$ encountered in example 115. We reproduce their graphs below in figures 5.1 through 5.3 for easy reference.

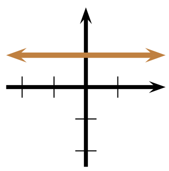

Figure 5.1: $x \mapsto 1$.

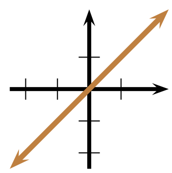

Figure 5.2: $x \mapsto x$.

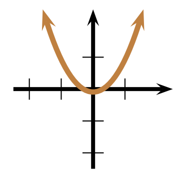

Figure 5.3: $x \mapsto x^2$.

The graphs above were obtained by geometrical arguments using similar triangles and the distance formula. This method of obtaining graphs of functions is quite limited, and hence, as a view of introducing a more general method that argues from the angles of continuity, monotonicity, and convexity, we will derive the shape of their graphs once more.

5.2 Affine Functions

255 Definition Let m, k be real number constants. A function of the form $x \mapsto mx + k$ is called an *affine function*. In the particular case that $m = 0$, we call $x \mapsto k$ a *constant function*. If, however, $k = 0$ and $m \neq 0$, then we call the function $x \mapsto mx$ a *linear function*.

256 Theorem (Graph of an Affine Function) The graph of an affine function

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & mx + k \end{array}$$

is a continuous straight line. It is strictly increasing if $m > 0$ and strictly decreasing if $m < 0$. If $m \neq 0$ then $x \mapsto mx + k$ has a unique zero $x = -\dfrac{k}{m}$. If $m \neq 0$ then $\mathbf{Im}(f) = \mathbb{R}$.

Proof: *Since for any $a \in \mathbb{R}$, $f(a+) = f(a) = f(a-) = ma + k$, an affine function is everywhere continuous. Let $\lambda \in [0;1]$. Since*

$$f(\lambda a + (1-\lambda)b) = m(\lambda a + (1-\lambda)b) + k = m\lambda a + mb - mb\lambda + k = \lambda m f(a) + (1-\lambda) m f(b),$$

an affine function is both convex and concave. This means that it does not bend upwards or downwards (or that it bends upwards and downwards!) always, and hence, it must be a straight line. Let $a < b$. Then

$$\frac{f(b) - f(a)}{b - a} = \frac{mb + k - ma - k}{b - a} = m,$$

which is strictly positive for $m > 0$ and strictly negative for $m < 0$. This means that f is a strictly increasing function for $m > 0$ and strictly decreasing for $m < 0$. Also given any $a \in \mathbb{R}$ we have

$$f(x) = a \implies mx + k = a \implies x = \frac{a - k}{m},$$

which is a real number as long as $m \neq 0$. Hence every real number is an image of f meaning that $\mathbf{Im}(f) = \mathbb{R}$. In particular, if $a = 0$, then $x = -\dfrac{k}{m}$ is the only solution to the equation $f(x) = 0$. Clearly, if $m = 0$, then $\mathbf{Im}(f) = \{k\}$.❑

This information is summarised in the following tables.

x	$-\infty$	$-\dfrac{k}{m}$	$+\infty$
$f(x) = mx+k$	$\nearrow$	0	$\nearrow$

Figure 5.4: Variation chart for $x \mapsto mx+k$, with $m > 0$.

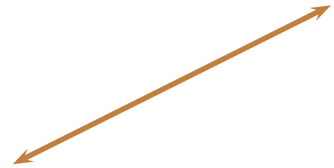

Figure 5.5: Graph of $x \mapsto mk+k$, $m > 0$.

x	$-\infty$	$-\dfrac{k}{m}$	$+\infty$
$f(x) = mx+k$	$\searrow$	0	$\searrow$

Figure 5.6: Variation chart for $x \mapsto mx+k$, with $m < 0$.

Figure 5.7: Graph of $x \mapsto mk+k$, $m < 0$.

Homework

5.2.1 Problem (Graph of the Absolute Value Function) Prove that the graph of the absolute value function

$$\mathbf{AbsVal}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & |x| \end{array}$$

is convex. Prove that $x \mapsto |x|$ is an even function, decreasing for $x < 0$ and increasing for $x > 0$. Moreover, prove that $\mathbf{Im}(\mathbf{AbsVal}) = [0; +\infty[$.

5.3 The Square Function

In this section we study the shape of the graph of the square function $x \mapsto x^2$.

257 Theorem (Graph of the Square Function) The graph of the square function

$$\mathbf{Sq}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is a convex curve which is strictly decreasing for $x < 0$ and strictly increasing for $x > 0$. Moreover, $x \mapsto x^2$ is an even function and $\mathbf{Im}(\mathbf{Sq}) = [0; +\infty[$.

Proof:

As $\mathbf{Sq}(-x) = (-x)^2 = x^2 = \mathbf{Sq}(x)$, *the square function is an even function. Now, for* $a < b$

$$\frac{\mathbf{Sq}(b) - \mathbf{Sq}(a)}{b-a} = \frac{b^2 - a^2}{b-a} = b + a.$$

If $a < b < 0$ *the sum* $a+b$ *is negative and* $x \mapsto x^2$ *is a strictly decreasing function. If* $0 < a < b$ *the sum* $a+b$ *is positive and* $x \mapsto x^2$ *is a strictly increasing function. To prove that* $x \mapsto x^2$ *is convex we observe that*

$$\begin{aligned}
& \mathbf{Sq}(\lambda a + (1-\lambda)b) \leq \lambda \mathbf{Sq}(a) + (1-\lambda)\mathbf{Sq}(b) \\
\iff \quad & \lambda^2 a^2 + 2\lambda(1-\lambda)ab + (1-\lambda)^2 b^2 \leq \lambda a^2 + (1-\lambda)b^2 \\
\iff \quad & 0 \leq \lambda(1-\lambda)a^2 - 2\lambda(1-\lambda)ab + ((1-\lambda) - (1-\lambda)^2)b^2 \\
\iff \quad & 0 \leq \lambda(1-\lambda)a^2 - 2\lambda(1-\lambda)ab + \lambda(1-\lambda)b^2 \\
\iff \quad & 0 \leq \lambda(1-\lambda)(a^2 - 2ab + b^2) \\
\iff \quad & 0 \leq \lambda(1-\lambda)(a-b)^2.
\end{aligned}$$

This last inequality is clearly true for $\lambda \in [0;1]$, *establishing the claim. Also suppose that* $y \in \mathbf{Im}(\mathbf{Sq})$. *Thus there is* $x \in \mathbb{R}$ *such that* $\mathbf{Sq}(x) = y \implies x^2 = y$. *But the equation* $y = x^2$ *is solvable only for* $y \geq 0$ *and so only positive numbers appear as the image of* $x \mapsto x^2$. *Since for* $x \in [0; +\infty[$ *we have* $\mathbf{Sq}(\sqrt{x}) = x$, *we conclude that* $\mathbf{Im}(\mathbf{Sq}) = [0; +\infty[$. *The graph of the* $x \mapsto x^2$ *is called a* parabola. *We summarise this information by means of the following diagram.*

x	$-\infty$		0		$+\infty$
$f(x) = x^2$		$\searrow$	0	$\nearrow$	

Figure 5.8: Variation chart for $x \mapsto x^2$.

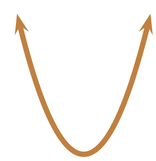

Figure 5.9: Graph of $x \mapsto x^2$.

❑

5.4 Quadratic Functions

258 Definition Let a, b, c be real numbers, with $a \neq 0$. A function of the form

$$f : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & ax^2 + bx + c \end{array}$$

is called a *quadratic function* with leading coefficient a.

259 Theorem Let $a \neq 0, b, c$ be real numbers and let $x \mapsto ax^2 + bx + c$ be a quadratic function. Then its graph is a parabola. If $a > 0$ the parabola has a local minimum at $x = -\dfrac{b}{2a}$ and it is convex. If $a < 0$ the parabola has a local maximum at $x = -\dfrac{b}{2a}$ and it is concave.

Proof: *Put $f(x) = ax^2 + bx + c$. Completing squares,*

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}, \end{aligned}$$

and hence this is a horizontal translation $-\dfrac{b}{2a}$ units and a vertical translation $\dfrac{4ac-b^2}{4a}$ units of the square function $x \mapsto x^2$ and so it follows from Theorems 257, 207 and 212, that the graph of f is a parabola.

Assume first that $a > 0$. Then f is convex, decreases if $x < -\dfrac{b}{2a}$ and increases if $x > -\dfrac{b}{2a}$, and so it has a minimum at $x = -\dfrac{b}{2a}$. The analysis of $-f$ yields the case for $a < 0$, and the Theorem is proved. ❑

The information of Theorem 259 is summarised in the following tables.

x	$-\infty$	$-\frac{b}{2a}$	$+\infty$
$f(x) = ax^2 + bx + c$	$\searrow$	0	$\nearrow$

Figure 5.10: $x \mapsto ax^2 + bx + c$, with $a > 0$.

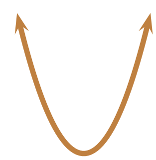

Figure 5.11: Graph of $x \mapsto ax^2 + bx + c$, $a > 0$.

x	$-\infty$	$-\frac{b}{2a}$	$+\infty$
$f(x) = ax^2 + bx + c$	$\nearrow$	0	$\searrow$

Figure 5.12: $x \mapsto ax^2 + bx + c$, with $a < 0$.

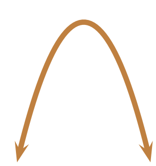

Figure 5.13: Graph of $x \mapsto ax^2 + bx + c$, $a < 0$.

260 Definition The point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ lies on the parabola and it is called the *vertex* of the parabola $y = ax^2+bx+c$. The quantity b^2-4ac is called the *discriminant* of ax^2+bx+c. The equation

$$y = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

is called the *canonical equation of the parabola* $y = ax^2+bx+c$.

The parabola $x \mapsto ax^2+bx+c$ is symmetric about the vertical line $x = -\frac{b}{2a}$ passing through its vertex. Notice that the axis of symmetry is parallel to the y-axis. If (h,k) is the vertex of the parabola, by completing squares, the equation of a parabola with axis of symmetry parallel to the y-axis can be written in the form $y = a(x-h)^2+k$. Using Theorem 107, the equation of a parabola with axis of symmetry parallel to the x-axis can be written in the form $x = a(y-k)^2+h$.

261 Example A parabola with axis of symmetry parallel to the y-axis and vertex at $(1,2)$. If the parabola passes through $(3,4)$, find its equation.

Solution: ► *The parabola has equation of the form $y = a(x-h)^2+k = a(x-1)^2+2$. Since when $x = 3$ we get $y = 4$, we have,*

$$4 = a(3-1)^2+2 \implies 4 = 4a+2 \implies a = \frac{1}{2}.$$

The equation sought is thus

$$y = \frac{1}{2}(x-1)^2+2.$$

◄

5.4.1 Zeros and Quadratic Formula

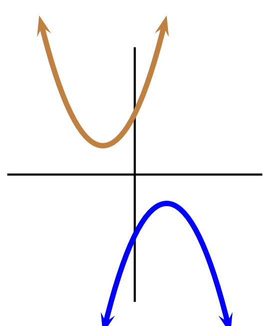
Figure 5.14: No real zeroes.

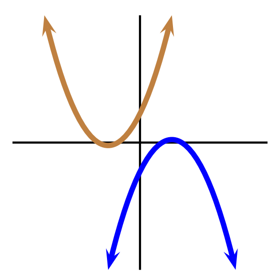
Figure 5.15: One real zero.

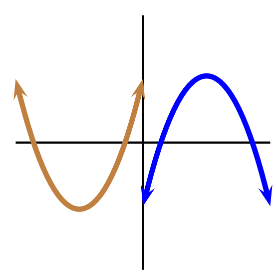
Figure 5.16: Two real zeros.

262 Definition In the quadratic equation $ax^2+bx+c = 0$, $a \neq 0$, the quantity b^2-4ac is called the *discriminant*.

263 Corollary (Quadratic Formula) The roots of the equation $ax^2+bx+c = 0$ are given by the formula

$$ax^2+bx+c = 0 \iff x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{5.1}$$

If $a \neq 0, b, c$ are real numbers and $b^2-4ac = 0$, the parabola $x \mapsto ax^2+bx+c$ is tangent to the x-axis and has one (repeated) real root. If $b^2-4ac > 0$ then the parabola has two distinct real roots. Finally, if $b^2-4ac < 0$ the parabola has two complex roots.

Proof: *By Theorem 259 we have*

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a},$$

and so

$$\begin{aligned} ax^2+bx+c=0 \quad &\iff \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \\ &\iff \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2|a|} \\ &\iff \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \end{aligned}$$

where we have dropped the absolute values on the last line because the only effect of having $a<0$ is to change from $\pm$ to $\mp$.

If $b^2-4ac=0$ then the vertex of the parabola is at $\left(-\frac{b}{2a},0\right)$ on the x-axis, and so the parabola is tangent there. Also, $x=-\frac{b}{2a}$ would be the only root of this equation. This is illustrated in figure 5.15.

If $b^2-4ac>0$, then $\sqrt{b^2-4ac}$ is a real number $\neq 0$ and so $\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $\frac{-b+\sqrt{b^2-4ac}}{2a}$ are distinct numbers. This is illustrated in figure 5.16.

If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ is a complex number $\neq 0$ and so $\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $\frac{-b+\sqrt{b^2-4ac}}{2a}$ are distinct complex numbers. This is illustrated in figure 5.14. ❑

If a quadratic has real roots, then the vertex lies on a line crossing the midpoint between the roots.

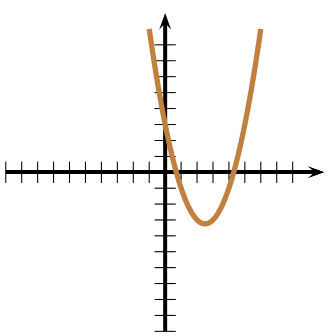

Figure 5.17: $y=x^2-5x+3$

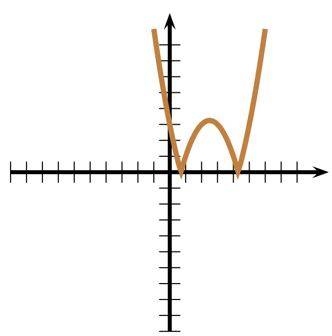

Figure 5.18: $y=|x^2-5x+3|$

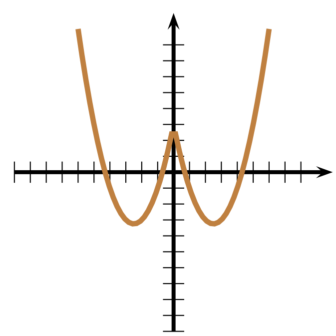

Figure 5.19: $y=|x|^2-5|x|+3$

264 Example Consider the quadratic function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2-5x+3$.

1. Write this parabola in canonical form and hence find the vertex of f. Determine the intervals of monotonicity of f and its convexity.
2. Find the x-intercepts and y-intercepts of f.
3. Graph $y=f(x)$, $y=|f(x)|$, and $y=f(|x|)$.
4. Determine the set of real numbers x for which $f(x)>0$.

Solution: ▶

1. *Completing squares*
$$y = x^2 - 5x + 3 = \left(x - \frac{5}{2}\right)^2 - \frac{13}{4}.$$
From this the vertex is at $\left(\frac{5}{2}, -\frac{13}{4}\right)$. *Since the leading coefficient of* f *is positive,* f *will be increasing for* $x > \frac{5}{2}$ *and it will be decreasing for* $x < \frac{5}{2}$ *and* f *is concave for all real values of* x.
2. *For* $x = 0$, $f(0) = 0^2 - 5 \cdot 0 + 3 = 3$, *and hence* $y = f(0) = 3$ *is the* y*-intercept. By the quadratic formula,*
$$f(x) = 0 \iff x^2 - 5x + 3 = 0 \iff x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} = \frac{5 \pm \sqrt{13}}{2}.$$
Observe that $\frac{5-\sqrt{13}}{2} \approx 0.697224362$ *and* $\frac{5+\sqrt{13}}{2} \approx 4.302775638$.
3. *The graphs appear in figures 5.17 through 5.19.*
4. *From the graph in figure 5.17,* $x^2 - 5x + 3 > 0$ *for values* $x \in \left]-\infty; \frac{5-\sqrt{13}}{2}\right[$ *or* $x \in \left]\frac{5+\sqrt{13}}{2}; +\infty\right[$.

◀

265 Corollary If $a \neq 0, b, c$ are real numbers and if $b^2 - 4ac < 0$, then $ax^2 + bx + c$ has the same sign as a.

Proof: *Since*
$$ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right),$$
and $4ac - b^2 > 0$, $\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right) > 0$ *and so* $ax^2 + bx + c$ *has the same sign as* a. ❑

266 Example Prove that the quantity $q(x) = 2x^2 + x + 1$ is positive regardless of the value of x.

Solution: ▶ *The discriminant is* $1^2 - 4(2)(1) = -7 < 0$, *hence the roots are complex. By Corollary 265, since its leading coefficient is* $2 > 0$, $q(x) > 0$ *regardless of the value of* x. *Another way of seeing this is to complete squares and notice the inequality*
$$2x^2 + x + 1 = 2\left(x + \frac{1}{4}\right)^2 + \frac{7}{8} \geq \frac{7}{8},$$
since $\left(x + \frac{1}{4}\right)^2$ *being the square of a real number, is* ≥ 0. ◀

By Corollary 263, if $a \neq 0, b, c$ are real numbers and if $b^2 - 4ac \neq 0$ then the numbers
$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
are distinct solutions of the equation $ax^2 + bx + c = 0$. Since
$$r_1 + r_2 = -\frac{b}{a}, \qquad \text{and} \qquad r_1 r_2 = \frac{c}{a},$$
any quadratic can be written in the form
$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right) = a\left(x^2 - (r_1 + r_2)x + r_1 r_2\right) = a(x - r_1)(x - r_2).$$
We call $a(x - r_1)(x - r_2)$ a *factorisation* of the quadratic $ax^2 + bx + c$.

267 Example A quadratic polynomial p has $1\pm\sqrt{5}$ as roots and it satisfies $p(1)=2$. Find its equation.

Solution: ► *Observe that the sum of the roots is*

$$r_1+r_2=1-\sqrt{5}+1+\sqrt{5}=2$$

and the product of the roots is

$$r_1r_2=(1-\sqrt{5})(1+\sqrt{5})=1-(\sqrt{5})^2=1-5=-4.$$ [1]

Hence p has the form

$$p(x)=a\left(x^2-(r_1+r_2)x+r_1r_2\right)=a(x^2-2x-4).$$

Since

$$2=p(1)\implies 2=a(1^2-2(1)-4)\implies a=-\frac{2}{5},$$

the polynomial sought is

$$p(x)=-\frac{2}{5}\left(x^2-2x-4\right).$$

◄

Homework

5.4.1 Problem Let

$$R_1=\{(x,y)\in\mathbb{R}^2|y\geq x^2-1\},$$

$$R_2=\{(x,y)\in\mathbb{R}^2|x^2+y^2\leq 4\},$$

$$R_3=\{(x,y)\in\mathbb{R}^2|y\leq -x^2+4\}.$$

Sketch the following regions.

1. $R_1\setminus R_2$
2. $R_1\cap R_3$
3. $R_2\setminus R_1$
4. $R_1\cap R_2$

5.4.2 Problem Write the following parabolas in canonical form, determine their vertices and graph them: (i) $y=x^2+6x+9$, (ii) $y=x^2+12x+35$, (iii) $y=(x-3)(x+5)$, (iv) $y=x(1-x)$, (v) $y=2x^2-12x+23$, (vi) $y=3x^2-2x+\frac{8}{9}$, (vii) $y=\frac{1}{5}x^2+2x+13$

5.4.3 Problem Find the vertex of the parabola $y=(3x-9)^2-9$.

5.4.4 Problem Find the equation of the parabola whose axis of symmetry is parallel to the y-axis, with vertex at $(0,-1)$ and passing through $(3,17)$.

5.4.5 Problem Find the equation of the parabola having roots at $x=-3$ and $x=4$ and passing through $(0,24)$.

5.4.6 Problem Let $0\leq a,b,c\leq 1$. Prove that at least one of the products $a(1-b),b(1-c),c(1-a)$ is smaller than or equal to $\frac{1}{4}$.

5.4.7 Problem An apartment building has 30 units. If all the units are inhabited, the rent for each unit is \$700 per unit. For every empty unit, management increases the rent of the remaining tenants by \$25. What will be the profit $P(x)$ that management gains when x units are empty? What is the maximum profit?

5.4.8 Problem Find all real solutions to $|x^2-2x|=|x^2+1|$.

5.4.9 Problem Find all the real solutions to

$$(x^2+2x-3)^2=2.$$

5.4.10 Problem Solve $x^3-x^2-9x+9=0$.

5.4.11 Problem Solve $x^3-2x^2-11x+12=0$.

5.4.12 Problem Find all real solutions to $x^3-1=0$.

5.4.13 Problem A parabola with axis of symmetry parallel to the x-axis and vertex at $(1,2)$. If the parabola passes through $(3,4)$, find its equation.

5.4.14 Problem Solve $9+x^{-4}=10x^{-2}$.

5.4.15 Problem Find all the real values of the parameter t for which the equation in x

$$t^2x-3t=81x-27$$

has a solution.

5.4.16 Problem The sum of two positive numbers is 50. Find the largest value of their product.

[1] As a shortcut for this multiplication you may wish to recall the *difference of squares identity*: $(a-b)(a+b)=a^2-b^2$.

5.4.17 Problem Of all rectangles having perimeter 20 shew that the square has the largest area.

5.4.18 Problem An orchard currently has 25 trees, which produce 600 fruits each. It is known that for each additional tree planted, the production of each tree diminishes by 15 fruits. Find:

1. the current fruit production of the orchard,
2. a formula for the production obtained from each tree upon planting x more trees,
3. a formula $P(x)$ for the production obtained from the orchard upon planting x more trees.
4. How many trees should be planted in order to yield maximum production?

5.5 $x \mapsto x^{2n+2}, n \in \mathbb{N}$

The graphs of $y = x^2$, $y = x^4$, $y = x^6$, etc., resemble one other. For $-1 \leq x \leq 1$, the higher the exponent, the flatter the graph (closer to the x-axis) will be, since

$$|x| < 1 \implies \cdots < x^6 < x^4 < x^2 < 1.$$

For $|x| \geq 1$, the higher the exponent, the steeper the graph will be since

$$|x| > 1 \implies \cdots > x^6 > x^4 > x^2 > 1.$$

We collect this information in the following theorem, of which we omit the proof.

268 Theorem Let $n \geq 2$ be an integer and $f(x) = x^n$. Then if n is even, f is convex, f is decreasing for $x < 0$, and f is increasing for $x > 0$. Also, $f(-\infty) = f(+\infty) = +\infty$.

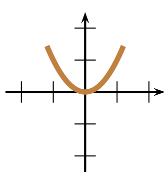

Figure 5.20: $y = x^2$.

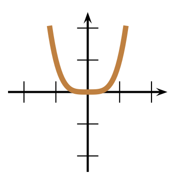

Figure 5.21: $y = x^4$.

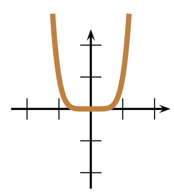

Figure 5.22: $y = x^6$.

x	$-\infty$		0		$+\infty$
$f(x) = x^n$		$\searrow$	0	$\nearrow$	

Figure 5.23: $x \mapsto x^n$, with $n > 0$ integer and even.

5.6 The Cubic Function

We now deduce properties for the cube function.

269 Theorem (Graph of the Cubic Function) The graph of the cubic function

$$\textbf{Cube}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is concave for $x < 0$ and convex for $x > 0$. $x \mapsto x^3$ is an increasing odd function and $\mathbf{Im}\left(\mathbf{Cube}\right) = \mathbb{R}$.

Proof: *Consider*

$$\mathbf{Cube}(\lambda a+(1-\lambda)b)-\lambda\mathbf{Cube}(a)-(1-\lambda)\mathbf{Cube}(b),$$

which is equivalent to

$$(\lambda a+(1-\lambda)b)^3-\lambda a^3-(1-\lambda)b^3,$$

which is equivalent to

$$(\lambda^3-\lambda)a^3+((1-\lambda)^3-(1-\lambda))b^3+3\lambda(1-\lambda)ab(\lambda a+(1-\lambda)b),$$

which is equivalent to

$$-(1-\lambda)(1+\lambda)\lambda a^3+(-\lambda^3+3\lambda^2-2\lambda)b^3+3\lambda(1-\lambda)ab(\lambda a+(1-\lambda)b),$$

which in turn is equivalent to

$$(1-\lambda)\lambda(-(1+\lambda)a^3+(\lambda-2)b^3+3ab(\lambda a+(1-\lambda)b)).$$

This last expression factorises as

$$-\lambda(1-\lambda)(a-b)^2(\lambda(a-b)+2b+a).$$

Since $\lambda(1-\lambda)(a-b)^2\geq 0$ *for* $\lambda\in[0;1]$,

$$\mathbf{Cube}(\lambda a+(1-\lambda)b)-\lambda\mathbf{Cube}(a)-(1-\lambda)\mathbf{Cube}(b)$$

has the same sign as

$$-(\lambda(a-b)+2b+a)=-(\lambda a+(1-\lambda)b+b+a).$$

If $(a,b)\in]0;+\infty[^2$ *then* $\lambda a+(1-\lambda)b\geq 0$ *by lemma 15 and so*

$$-(\lambda a+(1-\lambda)b+b+a)\leq 0$$

meaning that **Cube** *is convex for* $x\geq 0$. *Similarly, if* $(a,b)\in]-\infty;0[^2$ *then*

$$-(\lambda a+(1-\lambda)b+b+a)\geq 0$$

and so $x\mapsto x^3$ *is concave for* $x\geq 0$. *This proves the claim.*

As $\mathbf{Cube}(-x)=(-x)^3=-x^3=-\mathbf{Cube}(x)$, *the cubic function is an odd function. Since for* $a<b$

$$\frac{\mathbf{Cube}(b)-\mathbf{Cube}(a)}{b-a}=\frac{b^3-a^3}{b-a}=b^2+ab+b^2=\left(b+\frac{a}{2}\right)^2+\frac{3a^2}{4}>0,$$

Cube *is a strictly increasing function. Also if* $y\in\mathbf{Im}(\mathbf{Cube})$ *then there is* $x\in\mathbb{R}$ *such that* $x^3=\mathbf{Cube}(x)=y$. *The equation* $y=x^3$ *has a solution for every* $y\in\mathbb{R}$ *and so* $\mathbf{Im}(\mathbf{Cube})=\mathbb{R}$. *The graph of* $x\mapsto x^3$ *appears in figure 5.25.* ❑

5.7 $x\mapsto x^{2n+3}$, $n\in\mathbb{N}$

The graphs of $y=x^3$, $y=x^5$, $y=x^7$, etc., resemble one other. For $-1\leq x\leq 1$, the higher the exponent, the flatter the graph (closer to the x-axis) will be, since

$$|x|<1\implies\cdots<|x^7|<|x^5|<|x^3|<1.$$

For $|x|\geq 1$, the higher the exponent, the steeper the graph will be since

$$|x|>1\implies\cdots>|x^7|>|x^5|>|x^3|>1.$$

We collect this information in the following theorem, of which we omit the proof.

270 Theorem Let $n\geq 3$ be an integer and $f(x)=x^n$. Then if n is odd, f is increasing, f is concave for $x<0$, and f is convex for $x>0$. Also, $f(-\infty)=-\infty$ and $f(+\infty)=+\infty$.

x	$-\infty$	0	$+\infty$
$f(x) = x^n$	$\nearrow$	0	$\nearrow$

Figure 5.24: $x \mapsto x^n$, with $n \geq 3$ odd.

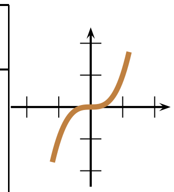

Figure 5.25: $y = x^3$.

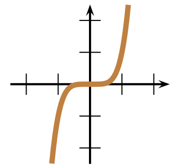

Figure 5.26: $y = x^5$.

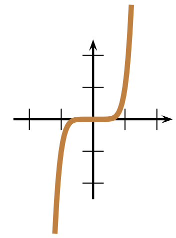

Figure 5.27: $y = x^7$.

5.8 Graphs of Polynomials

Recall that the zeroes of a polynomial $p(x) \in \mathbb{R}[x]$ are the solutions to the equation $p(x) = 0$, and that the polynomial is said to *split in* $\mathbb{R}$ if all the solutions to the equation $p(x) = 0$ are real.

In this section we study how to graph polynomials that split in $\mathbb{R}$, that is, we study how to graph polynomials of the form

$$p(x) = a(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k},$$

where $a \in \mathbb{R} \setminus \{0\}$ and the r_i are real numbers and the $m_i \geq 1$ are integers.

To graph such polynomials, we must investigate the *global behaviour* of the polynomial, that is, what happens as $x \to \pm\infty$, and we must also investigate the *local behaviour* around each of the roots r_i.

We start with the following theorem, which we will state without proof.

271 Theorem A polynomial function $x \mapsto p(x)$ is an everywhere continuous function.

272 Theorem Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0$, be a polynomial with real number coefficients. Then

$$p(-\infty) = (\text{signum}(a_n))(-1)^n\infty, \qquad p(+\infty) = (\text{signum}(a_n))\infty.$$

Thus a polynomial of odd degree will have opposite signs for values of large magnitude and different sign, and a polynomial of even degree will have the same sign for values of large magnitude and different sign.

Proof: *If $x \neq 0$ then*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \sim a_n x^n,$$

since as $x \to \pm\infty$, the quantity in parenthesis tends to 1 and so the eventual sign of $p(x)$ is determined by $a_n x^n$, which gives the result. ❑

We now state the basic result that we will use to graph polynomials.

273 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the graph of the polynomial

$$p(x) = a(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k},$$

- crosses the x-axis at $x = r_i$ if m_i is odd.
- is tangent to the x-axis at $x = r_i$ if m_i is even.

- has a convexity change at $x = r_i$ if $m_i \geq 3$ and m_i is odd.

Proof: *Since the local behaviour of $p(x)$ is that of $c(x - r_i)^{m_i}$ (where c is a real number constant) near r_i, the theorem follows at once from our work in section 5.1.* ❑

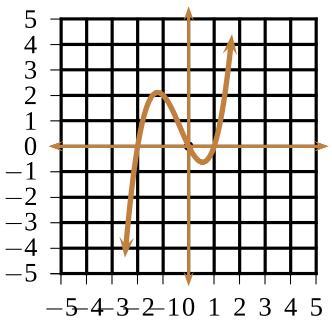

Figure 5.28: Example 274.

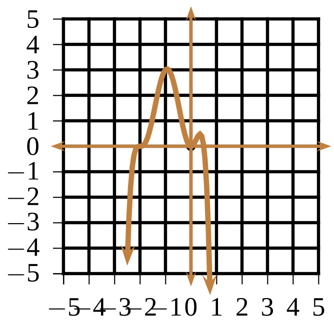

Figure 5.29: Example 275.

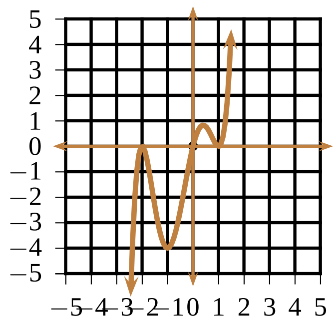

Figure 5.30: Example 276.

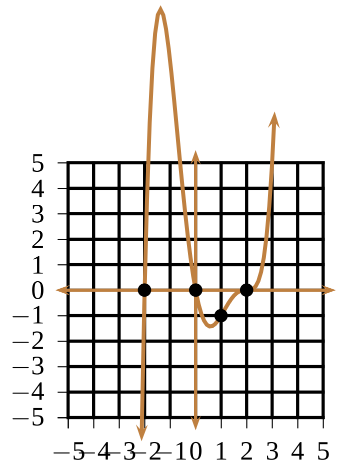

Figure 5.31: Example 277,.

274 Example Make a rough sketch of the graph of $y = (x+2)x(x-1)$. Determine where it achieves its local extrema and their values. Determine where it changes convexity.

Solution: ▶ *We have $p(x) = (x+2)x(x-1) \sim (x) \cdot x(x) = x^3$, as $x \to +\infty$. Hence $p(-\infty) = (-\infty)^3 = -\infty$ and $p(+\infty) = (+\infty)^3 = +\infty$. This means that for large negative values of x the graph will be on the negative side of the y-axis and that for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 273, the graph crosses the x-axis at $x = -2$, $x = 0$, and $x = 1$. The graph is shewn in figure 5.28.* ◀

275 Example Make a rough sketch of the graph of $y = (x+2)^3x^2(1-2x)$.

Solution: ▶ *We have $(x+2)^3x^2(1-2x) \sim x^3 \cdot x^2(-2x) = -2x^6$. Hence if $p(x) = (x+2)^3x^2(1-2x)$ then $p(-\infty) = -2(-\infty)^6 = -\infty$ and $p(+\infty) = -2(+\infty)^6 = -\infty$, which means that for both large positive and negative values of x the graph will be on the negative side of the y-axis. By Theorem 273, in a neighbourhood of $x = -2$, $p(x) \sim 20(x+2)^3$, so the graph crosses the x-axis changing convexity at $x = -2$. In a neighbourhood of 0, $p(x) \sim 8x^2$ and the graph is tangent to the x-axis at $x = 0$. In a neighbourhood of $x = \dfrac{1}{2}$, $p(x) \sim \dfrac{25}{16}(1-2x)$, and so the graph crosses the x-axis at $x = \frac{1}{2}$. The graph is shewn in figure 5.29.* ◀

276 Example Make a rough sketch of the graph of $y = (x+2)^2x(1-x)^2$.

Solution: ▶ *The dominant term of $(x+2)^2x(1-x)^2$ is $x^2 \cdot x(-x)^2 = x^5$. Hence if $p(x) = (x+2)^2x(1-x)^2$ then $p(-\infty) = (-\infty)^5 = -\infty$ and $p(+\infty) = (+\infty)^5 = +\infty$, which means that for large negative values of x the graph will be on the negative side of the y-axis and for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 273, the graph crosses the x-axis changing convexity at $x = -2$, it is tangent to the x-axis at $x = 0$ and it crosses the x-axis at $x = \frac{1}{2}$. The graph is shewn in figure 5.30.* ◀

277 Example , The polynomial in figure **??**, has degree 5. You may assume that the points marked below with a dot through which the polynomial passes have have integer coordinates. You may also assume that the graph of the polynomial changes concavity at $x = 2$.

1. Determine $p(1)$.
2. Find the general formula for $p(x)$.
3. Determine $p(3)$.

Solution: ►

1. *From the graph* $p(1) = -1$.
2. p *has roots at* $x = -2$, $x = 0$, $x = +2$. *Moreover,* p *has a zero of multiplicity at* $x = 2$*, and so it must have an equation of the form* $p(x) = A(x+2)(x)(x-2)^3$. *Now*

$$-1 = p(1) = A(1+2)(1)(1-2)^3 \implies A = \frac{1}{3} \implies p(x) = \frac{(x+2)(x)(x-2)^3}{3}.$$

3. $p(3) = \dfrac{(3+2)(3)(3-2)^3}{3} = 5.$

◄

Homework

5.8.1 Problem Make a rough sketch of the following curves.

1. $y = x^3 - x$
2. $y = x^3 - x^2$
3. $y = x^2(x-1)(x+1)$
4. $y = x(x-1)^2(x+1)^2$
5. $y = x^3(x-1)(x+1)$
6. $y = -x^2(x-1)^2(x+1)^3$
7. $y = x^4 - 8x^2 + 16$

5.8.2 Problem The polynomial in figure 5.32 has degree 4.

1. Determine $p(0)$.
2. Find the equation of $p(x)$.
3. Find $p(-3)$.
4. Find $p(2)$.

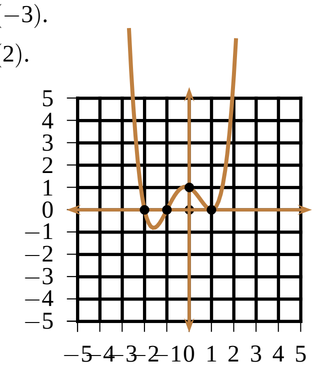

Figure 5.32: Problem 5.8.2.

5.9 Polynomials

5.9.1 Roots

In sections 5.2 and 5.4 we learned how to find the roots of equations (in the unknown x) of the type $ax+b=0$ and $ax^2+bx+c=0$, respectively. We would like to see what can be done for equations where the power of x is higher than 2. We recall that

278 Definition If all the roots of a polynomial are in $\mathbb{Z}$ (integer roots), then we say that the *polynomial splits or factors over* $\mathbb{Z}$. If all the roots of a polynomial are in $\mathbb{Q}$ (rational roots), then we say that the *polynomial splits or factors over* $\mathbb{Q}$. If all the roots of a polynomial are in $\mathbb{C}$ (complex roots), then we say that the *polynomial splits (factors) over* $\mathbb{C}$.

STOP *Since* $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$*, any polynomial splitting on a smaller set immediately splits over a larger set.*

279 Example The polynomial $l(x) = x^2 - 1 = (x-1)(x+1)$ splits over $\mathbb{Z}$. The polynomial $p(x) = 4x^2 - 1 = (2x-1)(2x+1)$ splits over $\mathbb{Q}$ but not over $\mathbb{Z}$. The polynomial $q(x) = x^2 - 2 = (x-\sqrt{2})(x+\sqrt{2})$ splits over $\mathbb{R}$ but not over $\mathbb{Q}$. The polynomial $r(x) = x^2 + 1 = (x-i)(x+i)$ splits over $\mathbb{C}$ but not over $\mathbb{R}$. Here $i = \sqrt{-1}$ is the imaginary unit.

5.9.2 Ruffini's Factor Theorem

280 Theorem (Division Algorithm) If the polynomial $p(x)$ is divided by $a(x)$ then there exist polynomials $q(x), r(x)$ with

$$p(x) = a(x)q(x) + r(x) \tag{5.2}$$

and $0 \leq$ degree $r(x) <$ degree $a(x)$.

281 Example If $x^5 + x^4 + 1$ is divided by $x^2 + 1$ we obtain

$$x^5 + x^4 + 1 = (x^3 + x^2 - x - 1)(x^2 + 1) + x + 2,$$

and so the quotient is $q(x) = x^3 + x^2 - x - 1$ and the remainder is $r(x) = x + 2$.

282 Example Find the remainder when $(x+3)^5 + (x+2)^8 + (5x+9)^{1997}$ is divided by $x + 2$.

Solution: ► *As we are dividing by a polynomial of degree* 1*, the remainder is a polynomial of degree 0, that is, a constant. Therefore, there is a polynomial* $q(x)$ *and a constant* r *with*

$$(x+3)^5 + (x+2)^8 + (5x+9)^{1997} = q(x)(x+2) + r$$

Letting $x = -2$ *we obtain*

$$(-2+3)^5 + (-2+2)^8 + (5(-2)+9)^{1997} = q(-2)(-2+2) + r = r.$$

As the sinistral side is 0 we deduce that the remainder $r = 0$. ◄

283 Example A polynomial leaves remainder -2 upon division by $x - 1$ and remainder -4 upon division by $x + 2$. Find the remainder when this polynomial is divided by $x^2 + x - 2$.

Solution: ► *From the given information, there exist polynomials* $q_1(x), q_2(x)$ *with* $p(x) = q_1(x)(x-1) - 2$ *and* $p(x) = q_2(x)(x+2) - 4$*. Thus* $p(1) = -2$ *and* $p(-2) = -4$*. As* $x^2 + x - 2 = (x-1)(x+2)$ *is a polynomial of degree 2, the remainder* $r(x)$ *upon dividing* $p(x)$ *by* $x^2 + x - 1$ *is of degree 1 or smaller, that is* $r(x) = ax + b$ *for some constants* a, b *which we must determine. By the Division Algorithm,*

$$p(x) = q(x)(x^2 + x - 1) + ax + b.$$

Hence

$$-2 = p(1) = a + b$$

and

$$-4 = p(-2) = -2a + b.$$

From these equations we deduce that $a = 2/3, b = -8/3$*. The remainder sought is*

$$r(x) = \frac{2}{3}x - \frac{8}{3}.$$

◄

284 Theorem (Ruffini's Factor Theorem) The polynomial $p(x)$ is divisible by $x - a$ if and only if $p(a) = 0$. Thus if p is a polynomial of degree n, then $p(a) = 0$ if and only if $p(x) = (x-a)q(x)$ for some polynomial q of degree $n - 1$.

Proof: *As* $x - a$ *is a polynomial of degree* 1*, the remainder after diving* $p(x)$ *by* $x - a$ *is a polynomial of degree* 0*, that is, a constant. Therefore*

$$p(x) = q(x)(x-a) + r.$$

From this we gather that $p(a) = q(a)(a-a) + r = r$*, from where the theorem easily follows.* ❑

285 Example Find the value of a so that the polynomial

$$t(x) = x^3 - 3ax^2 + 2$$

be divisible by $x+1$.

Solution: ► *By Ruffini's Theorem 284, we must have*

$$0 = t(-1) = (-1)^3 - 3a(-1)^2 + 2 \implies a = \frac{1}{3}.$$

◄

286 Definition Let a be a root of a polynomial p. We say that a is a root of *multiplicity* m if $p(x)$ is divisible by $(x-a)^m$ but not by $(x-a)^{m+1}$. This means that p can be written in the form $p(x) = (x-a)^m q(x)$ for some polynomial q with $q(a) \neq 0$.

287 Corollary If a polynomial of degree n had any roots at all, then it has at most n roots.

Proof: *If it had at least $n+1$ roots then it would have at least $n+1$ factors of degree 1 and hence degree $n+1$ at least, a contradiction.* ❑

Notice that the above theorem only says that if a polynomial has any roots, then it must have at most its degree number of roots. It does not say that a polynomial must possess a root. That all polynomials have at least one root is much more difficult to prove. We will quote the theorem, without a proof.

288 Theorem (Fundamental Theorem of Algebra) A polynomial of degree at least one with complex number coefficients has at least one complex root.

The Fundamental Theorem of Algebra implies then that a polynomial of degree n has exactly *n roots (counting multiplicity).*

A more useful form of Ruffini's Theorem is given in the following corollary.

289 Corollary If the polynomial p with integer coefficients,

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

has a rational root $\frac{s}{t} \in \mathbb{Q}$ (here $\frac{s}{t}$ is assumed to be in lowest terms), then s divides a_0 and t divides a_n.

Proof: *We are given that*

$$0 = p\left(\frac{s}{t}\right) = a_n\left(\frac{s^n}{t^n}\right) + a_{n-1}\left(\frac{s^{n-1}}{t^{n-1}}\right) + \cdots + a_1\left(\frac{s}{t}\right) + a_0.$$

Clearing denominators,

$$0 = a_n s^n + a_{n-1}s^{n-1}t + \cdots + a_1 s t^{n-1} + a_0 t^n.$$

This last equality implies that

$$-a_0 t^n = s(a_n s^{n-1} + a_{n-1}s^{n-2}t + \cdots + a_1 t^{n-1}).$$

Since both sides are integers, and since s and t have no factors in common, then s must divide a_0. We also gather that

$$-a_n s^n = t(a_{n-1}s^{n-1} + \cdots + a_1 s t^{n-2} + a_0 t^{n-1}),$$

from where we deduce that t divides a_n, concluding the proof. ❑

290 Example Factorise $a(x) = x^3 - 3x - 5x^2 + 15$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: ► *By Corollary 289, if* $a(x)$ *has integer roots then they must be in the set* $\{-1,1,-3,3,-5,5\}$. *We test* $a(\pm 1), a(\pm 3), a(\pm 5)$ *to see which ones vanish. We find that* $a(5) = 0$. *By the Factor Theorem,* $x-5$ *divides* $a(x)$. *Using long division,*

$$\begin{array}{r|l} & x^2 \qquad\quad -3 \\ \hline x-5 & x^3 - 5x^2 - 3x + 15 \\ & \underline{-x^3 + 5x^2} \\ & \qquad\quad -3x + 15 \\ & \qquad\quad \underline{3x - 15} \\ & \qquad\qquad\quad 0 \end{array}$$

we find

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x^2-3),$$

which is the required factorisation over $\mathbb{Z}[x]$. *The factorisation over* $\mathbb{R}[x]$ *is then*

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x-\sqrt{3})(x+\sqrt{3}).$$

◄

291 Example Factorise $b(x) = x^5 - x^4 - 4x + 4$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: ► *By Corollary 289, if* $b(x)$ *has integer roots then they must be in the set* $\{-1,1,-2,2,-4,4\}$. *We quickly see that* $b(1) = 0$, *and so, by the Factor Theorem,* $x-1$ *divides* $b(x)$. *By long division*

$$\begin{array}{r|l} & x^4 \qquad\quad -4 \\ \hline x-1 & x^5 - x^4 - 4x + 4 \\ & \underline{-x^5 + x^4} \\ & \qquad\quad -4x + 4 \\ & \qquad\quad \underline{4x - 4} \\ & \qquad\qquad\quad 0 \end{array}$$

we see that

$$b(x) = (x-1)(x^4-4) = (x-1)(x^2-2)(x^2+2),$$

which is the desired factorisation over $\mathbb{Z}[x]$. *The factorisation over* $\mathbb{R}$ *is seen to be*

$$b(x) = (x-1)(x-\sqrt{2})(x+\sqrt{2})(x^2+2).$$

Since the discriminant of x^2+2 *is* $-8 < 0$, x^2+2 *does not split over* $\mathbb{R}$. ◄

292 Lemma Complex roots of a polynomial with real coefficients occur in conjugate pairs, that is, if p is a polynomial with real coefficients and if $u+vi$ is a root of p, then its conjugate $u-vi$ is also a root for p. Here $i = \sqrt{-1}$ is the imaginary unit.

Proof: *Assume*

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

and that $p(u+vi)=0$. Since the conjugate of a real number is itself, and conjugation is multiplicative (Theorem 472), we have

$$\begin{aligned} 0 &= \overline{0} \\ &= \overline{p(u+vi)} \\ &= \overline{a_0+a_1(u+vi)+\cdots+a_n(u+vi)^n} \\ &= \overline{a_0}+\overline{a_1(u+vi)}+\cdots+\overline{a_n(u+vi)^n} \\ &= a_0+a_1(u-vi)+\cdots+a_n(u-vi)^n \\ &= p(u-vi), \end{aligned}$$

whence $u-vi$ is also a root. ❑

Since the complex pair root $u \pm vi$ would give the polynomial with real coefficients

$$(x-u-vi)(x-u+vi) = x^2-2ux+(u^2+v^2),$$

we deduce the following theorem.

293 Theorem Any polynomial with real coefficients can be factored in the form

$$A(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k}(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots(x^2+a_lx+b_l)^{n_l},$$

where each factor is distinct, the m_i, l_k are positive integers and A, r_i, a_i, b_i are real numbers.

Homework

5.9.1 Problem Find the cubic polynomial p having zeroes at $x=-1,2,3$ and satisfying $p(1)=-24$.

5.9.2 Problem How many cubic polynomials with leading coefficient -2 are there splitting in the set $\{1,2,3\}$?

5.9.3 Problem Find the cubic polynomial c having a root of $x=1$, a root of multiplicity 2 at $x=-3$ and satisfying $c(2)=10$.

5.9.4 Problem A cubic polynomial p with leading coefficient 1 satisfies $p(1)=1, p(2)=4, p(3)=9$. Find the value of $p(4)$.

5.9.5 Problem The polynomial $p(x)$ has integral coefficients and $p(x)=7$ for four different values of x. Shew that $p(x)$ never equals 14.

5.9.6 Problem Find the value of a so that the polynomial

$$t(x)=x^3-3ax^2+12$$

be divisible by $x+4$..

5.9.7 Problem Let $f(x)=x^4+x^3+x^2+x+1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

5.9.8 Problem If $p(x)$ is a cubic polynomial with $p(1)=1, p(2)=2, p(3)=3, p(4)=5$, find $p(6)$.

5.9.9 Problem The polynomial $p(x)$ satisfies $p(-x)=-p(x)$. When $p(x)$ is divided by $x-3$ the remainder is 6. Find the remainder when $p(x)$ is divided by x^2-9.

5.9.10 Problem Factorise $x^3+3x^2-4x+12$ over $\mathbb{Z}[x]$.

5.9.11 Problem Factorise $3x^4+13x^3-37x^2-117x+90$ over $\mathbb{Z}[x]$.

5.9.12 Problem Find a, b such that the polynomial x^3+6x^2+ax+b be divisible by the polynomial x^2+x-12.

5.9.13 Problem How many polynomials $p(x)$ of degree at least one and integer coefficients satisfy

$$16p(x^2)=(p(2x))^2,$$

for all real numbers x?

6 Rational Functions and Algebraic Functions

6.1 The Reciprocal Function

294 Definition Given a function f we write $f(-\infty)$ for the value that f may eventually approach for large (in absolute value) and negative inputs and $f(+\infty)$ for the value that f may eventually approach for large (in absolute value) and positive input. The line $y=b$ is a (horizontal) *asymptote* for the function f if either

$$f(-\infty)=b \qquad \text{or} \qquad f(+\infty)=b.$$

295 Definition Let $k>0$ be an integer. A function f has a *pole of order k* at the point $x=a$ if $(x-a)^{k-1}f(x)\to\pm\infty$ as $x\to a$, but $(x-a)^k f(x)$ as $x\to a$ is finite. Some authors prefer to use the term *vertical asymptote*, rather than pole.

296 Example Since $xf(x)=1$, $f(0-)=-\infty$, $f(0+)=+\infty$ for $f:\begin{array}{ccc}\mathbb{R}\setminus\{0\} & \to & \mathbb{R}\setminus\{0\}\\ x & \mapsto & \dfrac{1}{x}\end{array}$, f has a pole of order 1 at $x=0$.

297 Theorem (Graph of the Reciprocal Function) The graph of the reciprocal function

$$\mathbf{Rec}:\begin{array}{ccc}\mathbb{R}\setminus\{0\} & \to & \mathbb{R}\\ x & \mapsto & \dfrac{1}{x}\end{array}$$

is concave for $x<0$ and convex for $x>0$. $x\mapsto\dfrac{1}{x}$ is decreasing for $x<0$ and $x>0$. $x\mapsto\dfrac{1}{x}$ is an odd function and $\mathbf{Im}(\mathbf{Rec})=\mathbb{R}\setminus\{0\}$.

Proof: *Assume first that $0<a<b$ and that $\lambda\in[0;1]$. By the Arithmetic-Mean-Geometric-Mean Inequality, Theorem* **??**, *we deduce that*

$$\frac{a}{b}+\frac{b}{a}\geq 2.$$

Hence the product

$$\begin{aligned}(\lambda a+(1-\lambda)b)\left(\frac{\lambda}{a}+\frac{1-\lambda}{b}\right) &= \lambda^2+(1-\lambda)^2+\lambda(1-\lambda)\left(\frac{a}{b}+\frac{b}{a}\right)\\ &\geq \lambda^2+(1-\lambda)^2+2\lambda(1-\lambda)\\ &= (\lambda+1-\lambda)^2\\ &= 1.\end{aligned}$$

Thus for $0<a<b$ we have

$$\frac{1}{\lambda a+(1-\lambda)b}\leq\left(\frac{\lambda}{a}+\frac{1-\lambda}{b}\right)\implies \mathbf{Rec}(\lambda a+(1-\lambda)b)\leq\lambda\mathbf{Rec}(a)+(1-\lambda)\mathbf{Rec}(b),$$

from where $x\mapsto\dfrac{1}{x}$ is convex for $x>0$. If we replace a,b by $-a,-b$ then the inequality above is reversed and we obtain that $x\mapsto\dfrac{1}{x}$ is concave for $x<0$.

As $\mathbf{Rec}(-x) = \dfrac{1}{-x} = -\dfrac{1}{x} = -\mathbf{Rec}(x)$*, the reciprocal function is an odd function. Assume* $a < b$ *are non-zero and have the same sign. Then*

$$\frac{\mathbf{Rec}(b) - \mathbf{Rec}(a)}{b-a} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = -\frac{1}{ab} < 0,$$

since we are assuming that a, b *have the same sign, whence* $x \mapsto \dfrac{1}{x}$ *is a strictly decreasing function whenever arguments have the same sign. Also given any* $y \in \mathbf{Im}(\mathbf{Rec})$ *we have* $y = \mathbf{Rec}(x) = \dfrac{1}{x}$*, but this equation is solvable only if* $y \neq 0$*. and so every real number is an image of* $\mathbf{Id}$ *meaning that* $\mathbf{Im}(\mathbf{Rec}) = \mathbb{R} \setminus \{0\}$*.*

❑

298 Example Figures 6.1 through 6.3 exhibit various transformations of $y = a(x) = \dfrac{1}{x}$. Notice how the poles and the asymptotes move with the various transformations.

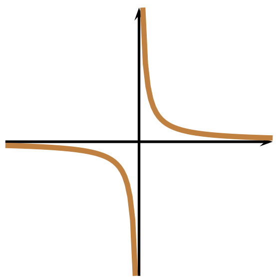

Figure 6.1: $x \mapsto \dfrac{1}{x}$

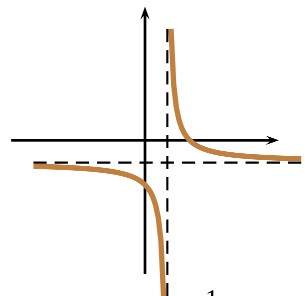

Figure 6.2: $x \mapsto \dfrac{1}{x-1} - 1$

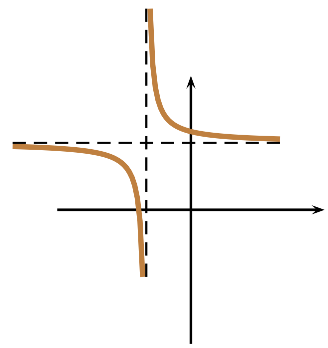

Figure 6.3: $x \mapsto \dfrac{1}{x+2} + 3$

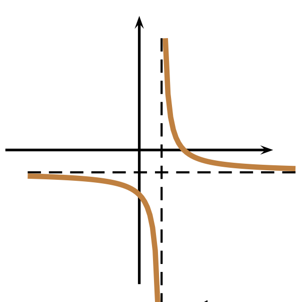

Figure 6.4: $x \mapsto \dfrac{1}{x-1} - 1$

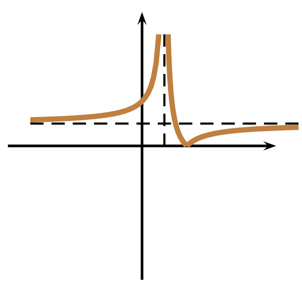

Figure 6.5: $x \mapsto \left|\dfrac{1}{x-1} - 1\right|$

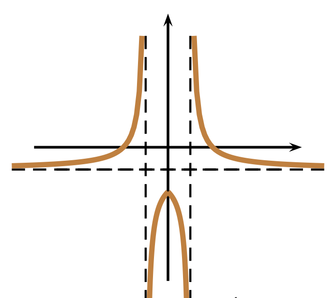

Figure 6.6: $x \mapsto \dfrac{1}{|x|-1} - 1$

6.2 Inverse Power Functions

We now proceed to investigate the behaviour of functions of the type $x \mapsto \dfrac{1}{x^n}$, where $n > 0$ is an integer.

299 Theorem Let $n > 0$ be an integer. Then

- if n is even, $x \mapsto \dfrac{1}{x^n}$ is increasing for $x < 0$, decreasing for $x > 0$ and convex for all $x \neq 0$.
- if n is odd, $x \mapsto \dfrac{1}{x^n}$ is decreasing for all $x \neq 0$, concave for $x < 0$, and convex for $x > 0$.

Thus $x \mapsto \dfrac{1}{x^n}$ has a pole of order n at $x = 0$ and a horizontal asymptote at $y = 0$.

300 Example A few functions $x \mapsto \dfrac{1}{x^n}$ are shewn in figures 6.7 through 6.12.

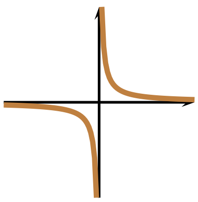

Figure 6.7: $x \mapsto \dfrac{1}{x}$

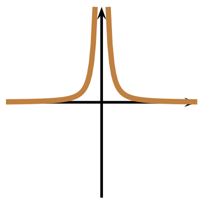

Figure 6.8: $x \mapsto \dfrac{1}{x^2}$

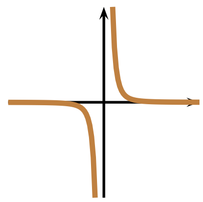

Figure 6.9: $x \mapsto \dfrac{1}{x^3}$

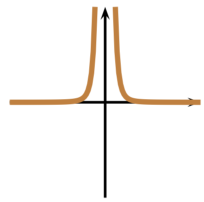

Figure 6.10: $x \mapsto \dfrac{1}{x^4}$

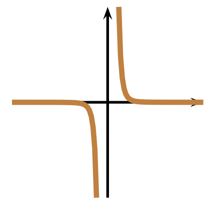

Figure 6.11: $x \mapsto \dfrac{1}{x^5}$

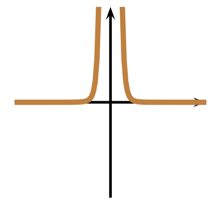

Figure 6.12: $x \mapsto \dfrac{1}{x^6}$

6.3 Rational Functions

301 Definition By a *rational function* $x \mapsto r(x)$ we mean a function r whose assignment rule is of the $r(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x) \neq 0$ are polynomials.

We now provide a few examples of graphing rational functions. Analogous to theorem 273, we now consider rational functions $x \mapsto r(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials with no factors in common and splitting in $\mathbb{R}$.

302 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the rational function with assignment rule

$$r(x) = K\frac{(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k}}{(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x-b_l)^{n_l}},$$

- has zeroes at $x = a_i$ and poles at $x = b_j$.
- crosses the x-axis at $x = a_i$ if m_i is odd.
- is tangent to the x-axis at $x = a_i$ if m_i is even.
- has a convexity change at $x = a_i$ if $m_i \geq 3$ and m_i is odd.
- both $r(b_j-)$ and $r(b_j+)$ blow to infinity. If n_i is even, then they have the same sign infinity: $r(b_i+) = r(b_i-) = +\infty$ or $r(b_i+) = r(b_i-) = -\infty$. If n_i is odd, then they have different sign infinity: $r(b_i+) = -r(b_i-) = +\infty$ or $r(b_i+) = -r(b_i-) = -\infty$.

Proof: *Since the local behaviour of* $r(x)$ *is that of* $c(x-r_i)^{t_i}$ *(where* c *is a real number constant) near* r_i*, the theorem follows at once from Theorem 268 and 299.* ❑

303 Example Draw a rough sketch of $x \mapsto \dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$.

Solution: ▶ *Put* $r(x) = \dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$*. By Theorem 302,* r *has zeroes at* $x = 1$*, and* $x = -2$*, and poles at* $x = -1$ *and* $x = 2$*. As* $x \to 1$*,* $r(x) \sim \dfrac{3}{2}(x-1)^2$*, hence the graph of* r *is tangent to the axes, and positive, around* $x = 2$*. As* $x \to -2$*,* $r(x) \sim -\dfrac{9}{16}(x+2)$*, hence the graph of* r *crosses the* x*-axis at* $x = -2$*, coming from positive* y*-values on the left of* $x = -2$ *and going to negative* y*=values on the right of* $x = -2$*. As* $x \to -1$*,* $r(x) \sim \dfrac{4}{9(x+1)}$*, hence the* *graph of r blows to $-\infty$ to the left of $x=-1$ and to $+\infty$ to the right of $x=-1$. As $x \to 2$, $r(x) \sim \dfrac{4}{3(x-2)^2}$, hence the graph of r blows to $+\infty$ both from the left and the right of $x=2$. Also we observe that*

$$r(x) \sim \frac{(x)^2(x)}{(x)(x)^2} = \frac{x^3}{x^3} = 1,$$

and hence r has the horizontal asymptote $y=1$. A sign diagram for $\dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$ follows:

$]-\infty;-2[$	$]-2;-1[$	$]-1;1[$	$]1;2[$	$]2;+\infty[$
$+$	$-$	$+$	$+$	$+$

The graph of r can be found in figure 6.13. ◄

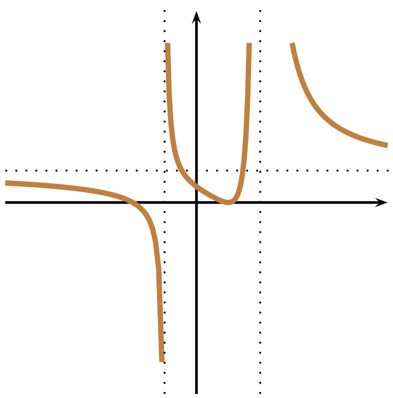

Figure 6.13: $x \mapsto \dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$

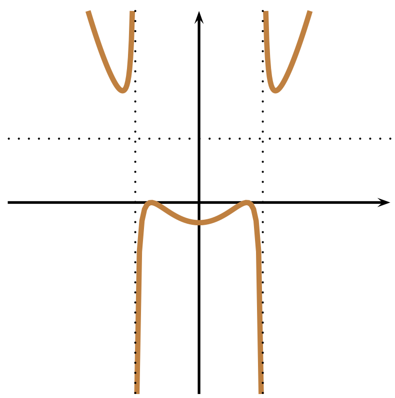

Figure 6.14: $x \mapsto \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$

304 Example Draw a rough sketch of $x \mapsto \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$.

Solution: ► *Put $r(x) = \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$. First observe that $r(x) = r(-x)$, and so r is even. By Theorem 302, r has zeroes at $x = \pm\dfrac{3}{4}$, and poles at $x = \pm 1$. As $x \to \dfrac{3}{4}$, $r(x) \sim -\dfrac{36}{7}(x-3/4)^2$, hence the graph of r is tangent to the axes, and negative, around $x = 3/4$, and similar behaviour occurs around $x = -\dfrac{3}{4}$. As $x \to 1$, $r(x) \sim \dfrac{49}{512(x-1)}$, hence the graph of r blows to $-\infty$ to the left of $x=1$ and to $+\infty$ to the right of $x=1$. As $x \to -1$, $r(x) \sim -\dfrac{49}{512(x-1)}$, hence the graph of r blows to $+\infty$ to the left of $x=-1$ and to $-\infty$ to the right of $x=-1$. Also, as $x \to +\infty$,*

$$r(x) \sim \frac{(x)^2(x)^2}{(x)(x)} = x^2,$$

so $r(+\infty) = +\infty$ *and* $r(-\infty) = +\infty$. *A sign diagram for* $\dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$ *follows:*

$]-\infty;-1[$	$\left]-1;-\dfrac{3}{4}\right[$	$\left]-\dfrac{3}{4};\dfrac{3}{4}\right[$	$\left]\dfrac{3}{4};1\right[$	$]1;+\infty[$
$+$	$-$	$-$	$-$	$+$

The graph of r can be found in figure 6.14. ◀

Homework

6.3.1 Problem Find the condition on the distinct real numbers a, b, c such that the function $x \mapsto \dfrac{(x-a)(x-b)}{x-c}$ takes all real values for real values of x. Sketch two scenarios to illustrate a case when the condition is satisfied and a case when the condition is not satisfied.

6.3.2 Problem Make a rough sketch of the following curves.

1. $y = \dfrac{x}{x^2-1}$
2. $y = \dfrac{x^2}{x^2-1}$
3. $y = \dfrac{x^2-1}{x}$
4. $y = \dfrac{x^2-x-6}{x^2+x-6}$
5. $y = \dfrac{x^2+x-6}{x^2-x-6}$
6. $y = \dfrac{x}{(x+1)^2(x-1)^2}$
7. $y = \dfrac{x^2}{(x+1)^2(x-1)^2}$

6.3.3 Problem The rational function q in figure 6.15 has only two simple poles and satisfies $q(x) \to 1$ as $x \to \pm\infty$. You may assume that the poles and zeroes of q are located at integer points.

1. Find $q(0)$.
2. Find $q(x)$ for arbitrary x.
3. Find $q(-3)$.
4. To which value does $q(x)$ approach as $x \to -2+$?

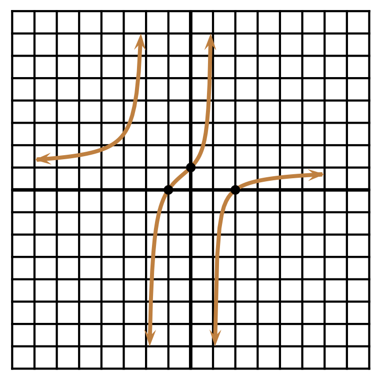

Figure 6.15: Problem 6.3.3.

6.4 Algebraic Functions

305 Definition We will call *algebraic function* a function whose assignment rule can be obtained from a rational function by a finite combination of additions, subtractions, multiplications, divisions, exponentiations to a rational power.

306 Theorem Let $|q| \geq 2$ be an integer. If

- if q is even then $x \mapsto x^{1/q}$ is increasing and concave for $q \geq 2$ and decreasing and convex for $q \leq -2$ for all $x > 0$ and it is undefined for $x < 0$.
- if q is odd then $x \mapsto x^{1/q}$ is everywhere increasing and convex for $x < 0$ but concave for $x > 0$ if $q \geq 3$. If $q \leq -3$ then $x \mapsto x^{1/q}$ is decreasing and concave for $x < 0$ and increasing and convex for $x > 0$.

A few of the functions $x \mapsto x^{1/q}$ are shewn in figures 6.16 through 6.27.

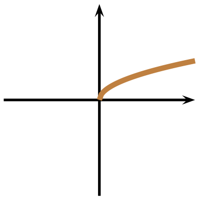

Figure 6.16: $x \mapsto x^{1/2}$

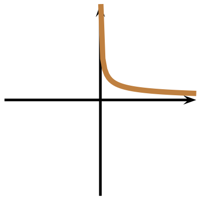

Figure 6.17: $x \mapsto x^{-1/2}$

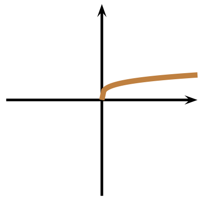

Figure 6.18: $x \mapsto x^{1/4}$

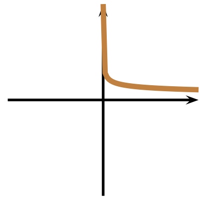

Figure 6.19: $x \mapsto x^{-1/4}$

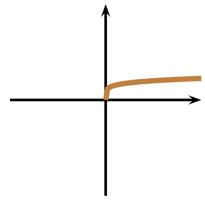

Figure 6.20: $x \mapsto x^{1/6}$

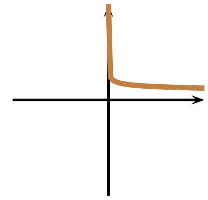

Figure 6.21: $x \mapsto x^{-1/6}$

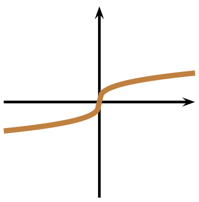

Figure 6.22: $x \mapsto x^{1/3}$

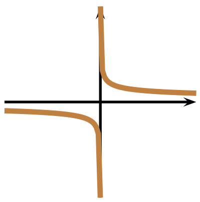

Figure 6.23: $x \mapsto x^{-1/3}$

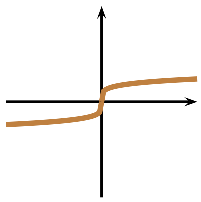

Figure 6.24: $x \mapsto x^{1/5}$

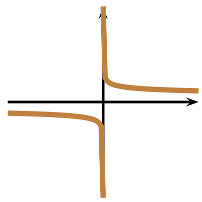

Figure 6.25: $x \mapsto x^{-1/5}$

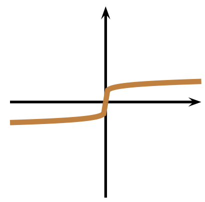

Figure 6.26: $x \mapsto x^{1/7}$

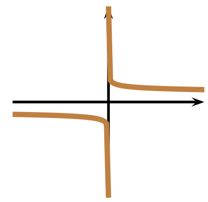

Figure 6.27: $x \mapsto x^{-1/7}$

Homework

6.4.1 Problem Draw the graph of each of the following curves.

1. $x \mapsto (1+x)^{1/2}$
2. $x \mapsto (1-x)^{1/2}$
3. $x \mapsto 1+(1+x)^{1/3}$
4. $x \mapsto 1-(1-x)^{1/3}$
5. $x \mapsto \sqrt{x}+\sqrt{-x}$

7 Exponential Functions

7.1 Exponential Functions

307 Definition Let $a > 0, a \neq 1$ be a fixed real number. The function

$$\begin{array}{rcl} \mathbb{R} & \to &]0;+\infty[\\ x & \mapsto & a^x \end{array},$$

is called the *exponential function* of *base a*.

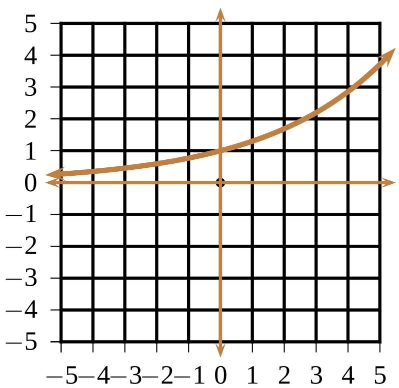

Figure 7.1: $x \mapsto a^x,\ a > 1$.

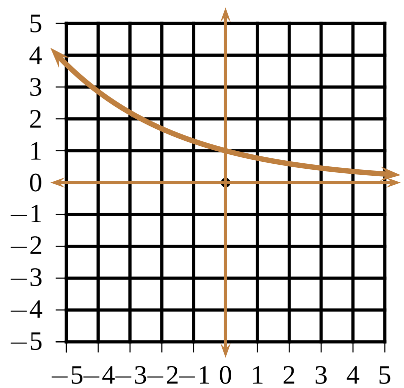

Figure 7.2: $x \mapsto a^x,\ 0 < a < 1$.

We will now prove that the generic graphs of the exponential function resemble those in figures 7.1 and 7.2.

308 Theorem If $a > 1$, $x \mapsto a^x$ is strictly increasing and convex, and if $0 < a < 1$ then $x \mapsto a^x$ is strictly decreasing and convex.

Proof: *Put $f(x) = a^x$. Recall that a function f is strictly increasing or decreasing depending on whether the ratio*

$$\frac{f(t) - f(s)}{t - s} > 0 \quad \text{or} \quad < 0$$

for $t \neq s$. Now,

$$\frac{f(t) - f(s)}{t - s} = \frac{a^t - a^s}{t - s} = (a^s) \cdot \frac{a^{t-s} - 1}{t - s}.$$

If $a > 1$, and $t - s > 0$ then also $a^{t-s} > 1$.[1] *If $a > 1$, and $t - s < 0$ then also $a^{t-s} < 1$. Thus regardless on whether $t - s > 0$ or < 0 the ratio*

$$\frac{a^{t-s} - 1}{t - s} > 0,$$

[1]The alert reader will find this argument circular! I have tried to prove this theorem from first principles without introducing too many tools. Alas, I feel tired. . .

whence f is increasing for $a>1$. A similar argument proves that for $0<a<1$, f would be decreasing.

To prove convexity will be somewhat more arduous. Recall that f is convex if for arbitrary $0\leq\lambda\leq 1$ we have

$$f(\lambda s+(1-\lambda)t)\leq\lambda f(s)+(1-\lambda)f(t),$$

that is, a straight line joining any two points of the curve lies above the curve. We will not be able to prove this quickly, we will just content with proving midpoint convexity: we will prove that

$$f\left(\frac{s+t}{2}\right)\leq\frac{1}{2}f(s)+\frac{1}{2}f(t).$$

This is equivalent to

$$a^{\frac{s+t}{2}}\leq\frac{1}{2}a^s+\frac{1}{2}a^t,$$

which in turn is equivalent to

$$2\leq a^{\frac{s-t}{2}}+a^{\frac{t-s}{2}}.$$

But the square of a real number is always non-negative, hence

$$\left(a^{\frac{s-t}{4}}-a^{\frac{t-s}{4}}\right)^2\geq 0\implies a^{\frac{s-t}{2}}+a^{\frac{t-s}{2}}\geq 2,$$

proving midpoint convexity. ❑

STOP *The line $y=0$ is an asymptote for $x\mapsto a^x$. If $a>1$, then $a^x\to 0$ as $x\to-\infty$ and $a^x\to+\infty$ as $x\to+\infty$. If $0<a<1$, then $a^x\to+\infty$ as $x\to-\infty$ and $a^x\to 0$ as $x\to+\infty$.*

Homework

7.1.1 Problem Make rough sketches of the following curves.

1. $x\mapsto 2^x$
2. $x\mapsto 2^{|x|}$
3. $x\mapsto 2^{-|x|}$
4. $x\mapsto 2^x+3$
5. $x\mapsto 2^{x+3}$

7.2 The number e

Consider now the following problem, first studied by the Swiss mathematician Jakob Bernoulli around the 1700s: Query: If a creditor lends money at interest under the condition that during each individual moment the proportional part of the annual interest be added to the principal, what is the balance at the end of a full year?[2]

Suppose a dollars are deposited, and the interest is added n times a year at a rate of x. After the first time period, the balance is

$$b_1=\left(1+\frac{x}{n}\right)a.$$

After the second time period, the balance is

$$b_2=\left(1+\frac{x}{n}\right)b_1=\left(1+\frac{x}{n}\right)^2a.$$

Proceeding recursively, after the n-th time period, the balance will be

$$b_n=\left(1+\frac{x}{n}\right)^na.$$

The study of the sequence

$$e_n=\left(1+\frac{1}{n}\right)^n$$

[2] "Quæritur, si creditor aliquis pecuniam suam fœnori exponat, ea lege, ut singulis momentis pars proportionalis usuræ annuæ sorti annumeretur; quantum ipsi finito anno debeatur?"

thus becomes important. It was Bernoulli's pupil, Leonhard Euler, who shewed that the sequence $\left(1+\frac{1}{n}\right)^n, n=1,2,3,\ldots$ converges to a finite number, which he called e. In other words, Euler shewed that

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \tag{7.1}$$

It must be said, in passing, that Euler did not rigourously shewed the existence of the above limit. He, however, gave other formulations of the irrational number

$$e=2.718281828459045235360287471352\ldots,$$

among others, the infinite series

$$e=2+\frac{1}{2!}++\frac{1}{3!}++\frac{1}{4!}++\frac{1}{5!}+\cdots, \tag{7.2}$$

and the infinite continued fraction

$$e=2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{\cdots}}}}}}}}}. \tag{7.3}$$

We will now establish a series of results in order to prove that the limit in 7.1 exists.

309 Lemma Let n be a positive integer. Then

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}).$$

Proof: *The lemma follows by direct multiplication of the dextral side.* ❑

310 Lemma If $0\leq a<b,\ n\in\mathbb{N}$

$$na^{n-1}<\frac{b^n-a^n}{b-a}<nb^{n-1}.$$

Proof: *By Lemma 309*

$$\begin{aligned}\frac{b^n-a^n}{b-a} &= b^{n-1}+b^{n-2}a+b^{n-3}a^2+\cdots+b^2a^{n-3}+ba^{n-2}+a^{n-1}\\ &< b^{n-1}+b^{n-1}+\cdots+b^{n-1}+b^{n-1}\\ &= nb^{n-1},\end{aligned}$$

from where the dextral inequality follows. The sinistral inequality can be established similarly. ❑

311 Theorem The sequence

$$e_n=\left(1+\frac{1}{n}\right)^n, n=1,2,\ldots$$

is a bounded increasing sequence, and hence it converges to a limit, which we call e.

Proof: *By Lemma 310*

$$\frac{b^{n+1}-a^{n+1}}{b-a} \leq (n+1)b^n \implies b^n[(n+1)a-nb] < a^{n+1}.$$

Putting $a = 1+\dfrac{1}{n+1}$, $b = 1+\dfrac{1}{n}$ *we obtain*

$$e_n = \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} = e_{n+1},$$

whence the sequence $e_n, n = 1,2,\ldots$ *increases. Again, by putting* $a = 1$, $b = 1+\dfrac{1}{2n}$ *we obtain*

$$\left(1+\frac{1}{2n}\right)^n < 2 \implies \left(1+\frac{1}{2n}\right)^{2n} < 4 \implies e_{2n} < 4.$$

Since $e_n < e_{2n} < 4$ *for all* n*, the sequence is bounded. In view of Theorem 515 the sequence converges to a limit. We call this limit* e*.* ❑

Since the sequence increases towards e *we have*

$$2 = \left(1+\frac{1}{1}\right)^1 < e.$$

From the proof of Theorem 311 it stems that $2 < e < 4$*. In fact, in can be shewn that* $e \approx 2.71828182845904523536028747135 2\ldots$ *and so* $2 < e < 3$*.*

e *is called the* natural *exponential base. The function* $x \mapsto e^x$ *has the property that any tangent drawn to the curve at the point* x *has slope* e^x*. The notation* $\exp(x) = \exp x = e^x$ *is often used.*

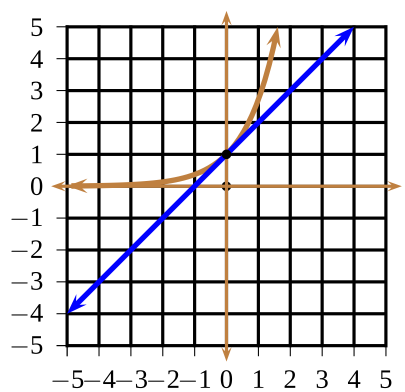

Figure 7.3: $1+x \leq e^x \ \forall x \in \mathbb{R}$.

312 Theorem If $x \in \mathbb{R}$ then

$$1+x \leq e^x,$$

with equality only for $x = 0$.

Proof: *From figure 7.3, the line* $y = 1+x$ *lies below the graph of* $y = e^x$*, proving the theorem.*❑

Replacing x by $x-1$ we obtain,

313 Corollary

$$x \leq e^{x-1}, \ \forall x \in \mathbb{R}.$$

Equality occurs if and only if $x = 1$.

Homework

7.2.1 Problem True or False.

1. $\exists t \in \mathbb{R}$ such that $e^t = -9$.
2. As $x \to -\infty$, $2^x \to -\infty$.
3. $\forall x \in \mathbb{R}$, $10 + x^2 + x^4 > 2^x$.
4. $x \mapsto e^x$ is increasing over $\mathbb{R}$.
5. $x \mapsto \frac{e^x}{\pi^x}$ is increasing over $\mathbb{R}$.
6. $ex \leq e^x$, $\forall x \in \mathbb{R}$.

7.2.2 Problem By using Theorem 312, and the fact that $\pi > e$, prove that $e^\pi > \pi^e$.

(Hint: Put $x = \frac{\pi}{e} - 1$.)

7.2.3 Problem Make a rough sketch of each of the following.

1. $x \mapsto 2^x$
2. $x \mapsto e^x$
3. $x \mapsto \left(\frac{1}{2}\right)^x$
4. $x \mapsto -1 + 2^x$
5. $x \mapsto e^{|x|}$
6. $x \mapsto e^{-|x|}$

7.2.4 Problem Let $n \in \mathbb{N}, n > 1$. Prove that

$$n! < \left(\frac{n+1}{2}\right)^n.$$

7.2.5 Problem Put

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

The function $x \mapsto \cosh x$ is known as the *hyperbolic cosine*. The function $x \mapsto \sinh x$ is known as the *hyperbolic sine*.

7.2.6 Problem Prove that for $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

and

$$\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{n+1}\right)^{n+2}.$$

(Hint: Use a suitable choice of a and b in Lemma 310.)

7.2.7 Problem Prove that the function $x \mapsto \dfrac{x}{e^x - 1} + \dfrac{x}{2}$ is even.

7.3 Arithmetic Mean-Geometric Mean Inequality

Using Corollary 313, we may prove, à la Pólya, the Arithmetic-Mean-Geometric-Mean Inequality (AM-GM Inequality, for short).

314 Theorem (Arithmetic-Mean-Geometric-Mean Inequality) Let

$$a_1, a_2, \ldots, a_n$$

be non-negative real numbers. Then

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Equality occurs if and only if $a_1 = a_2 = \ldots = a_n$.

Proof: *Put*

$$A_k = \frac{na_k}{a_1 + a_2 + \cdots + a_n},$$

and $G_n = a_1 a_2 \cdots a_n$. *Observe that*

$$A_1 A_2 \cdots A_n = \frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n},$$

and that

$$A_1 + A_2 + \cdots + A_n = n.$$

By Corollary 313, we have

$$A_1 \leq \exp(A_1 - 1),$$

$$A_2 \le \exp(A_2 - 1),$$
$$\vdots$$
$$A_n \le \exp(A_n - 1).$$

Since all the quantities involved are non-negative, we may multiply all these inequalities together, to obtain,

$$A_1 A_2 \cdots A_n \le \exp(A_1 + A_2 + \cdots + A_n - n).$$

In view of the observations above, the preceding inequality is equivalent to

$$\frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n} \le \exp(n - n) = e^0 = 1.$$

We deduce that

$$G_n \le \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n,$$

which is equivalent to

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now, for equality to occur, we need each of the inequalities $A_k \le \exp(A_k - 1)$ to hold. This occurs, in view of Corollary 313 if and only if $A_k = 1$, $\forall k$, which translates into $a_1 = a_2 = \ldots = a_n$.. This completes the proof. ❑

315 Corollary (Harmonic-Mean-Geometric-Mean Inequality) If

$$a_1, a_2, \ldots, a_n$$

are positive real numbers, then

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \le \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Proof: *By the AM-GM Inequality,*

$$\sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}} \le \frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n},$$

from where the result follows by rearranging. ❑

316 Example The sum of two positive real numbers is 100. Find their maximum product.

Solution: ► *Let x and y be the numbers. We use the AM-GM Inequality for $n = 2$. Then*

$$\sqrt{xy} \le \frac{x+y}{2}.$$

In our case, $x + y = 100$, and so

$$\sqrt{xy} \le 50,$$

which means that the maximum product is $xy \le 50^2 = 2500$. If we take $x = y = 50$, we see that the maximum product is achieved for this choice of x and y. ◄

317 Example From a rectangular cardboard piece measuring 75×45 a square of side x is cut from each of its corners in order to make an open box. See figure 7.4. Find the function $x \mapsto V(x)$ that gives the volume of the box as a function of x, and obtain an upper bound for the volume of this box.

Solution: ► *From the diagram shewn, the height of the box is x, its length $75-2x$ and its width $45-2x$. Hence*

$$V(x) = x(75-2x)(45-2x).$$

Now, if we used the AM-GM Inequality for the three quantities $x,\ 80-2x$, and $50-2x$, we would obtain

$$\begin{array}{lll} V(x) & = & x(75-2x)(45-2x) \\ & < & \left(\dfrac{x+75-2x+45-2x}{3}\right)^3 \\ & = & \left(\dfrac{120-3x}{3}\right)^3 \\ & = & (40-x)^3. \end{array}$$

(We use the strict inequality sign because we know that equality will never be achieved: $75-2x$ never equals $45-2x$.) This has the disadvantage of depending on x. In order to overcome this, we use the following trick. Consider, rather, the three quantities $4x,\ 75-2x$, and $45-2x$. Then

$$\begin{array}{lll} 4V(x) & = & 4x(75-2x)(45-2x) \\ & < & \left(\dfrac{4x+75-2x+45-2x}{3}\right)^3 \\ & = & \left(\dfrac{120}{3}\right)^3 \\ & = & 64000. \end{array}$$

This means that

$$V(x) < \frac{64000}{4} = 16000.$$

◄

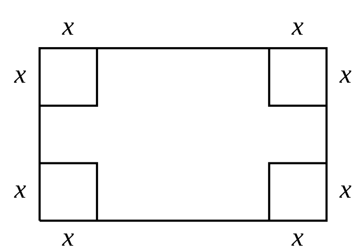

Figure 7.4: Example 317.

Later in calculus, you will see that the volume is

$$V(x) \leq (20-\frac{5}{2}\sqrt{19})(35+5\sqrt{19})(5+5\sqrt{19}),$$

and that the maximum is achieved when

$$x = 20-\frac{5}{2}\sqrt{19}.$$

318 Example Find the maximum value of the function $f : \begin{array}{lll} [0;1] & \to & \mathbb{R} \\ x & \mapsto & x(1-x)^2 \end{array}$.

Solution: ► *Observe that for $x \in [0;1]$ both x and $1-x$ are non-negative. We maximise, rather, $2f(x)$ via the AM-GM Inequality.*

$$2f(x) = 2x(1-x)^2 = 2x(1-x)(1-x) \leq \left(\frac{2x+1-x+1-x}{3}\right)^3 = \frac{8}{27}.$$

Thus

$$f(x) \leq \frac{1}{2} \cdot \frac{8}{27} = \frac{4}{27}.$$

The maximum value is attained when $2x = 1-x$, that is, when $x = \frac{1}{3}$. ◄

Homework

7.3.1 Problem Let x, y, z be any real numbers. Prove that

$$3x^2y^2z^2 \leq x^6 + y^6 + z^6.$$

7.3.2 Problem The sum of 5 positive real numbers is S. What is their maximum product?

7.3.3 Problem Use AM-GM to prove that $\cosh x \geq 1, \quad \forall x \in \mathbb{R}$.

7.3.4 Problem Maximise the following functions over $[0;1]$.

1. $a : x \mapsto x(1-x)^3$.
2. $b : x \mapsto x^2(1-x)^2$.
3. $c : x \mapsto x^2(1-x)^3$.

7.3.5 Problem Prove that of all rectangular boxes with a given surface area, the cube has the largest volume.

8 Logarithmic Functions

8.1 Logarithms

Recall that if $a > 0, a \neq 1$ is a fixed real number, $\begin{array}{rcl} \mathbb{R} & \to &]0;+\infty[\\ x & \mapsto & a^x \end{array}$ maps a real number x to a positive number y, i.e., $a^x = y$.

We call x the *logarithm* of y to the base a, and we write $x = \log_a y$. In other words, the function $\begin{array}{rcl} \mathbb{R} & \to &]0;+\infty[\\ x & \mapsto & a^x \end{array}$ has inverse $\begin{array}{rcl}]0;+\infty[& \to & \mathbb{R} \\ x & \mapsto & \log_a x \end{array}$.[1]

319 Example $\log_5 25 = 2$ since $5^2 = 25$.

320 Example $\log_2 1024 = 10$ since $2^{10} = 1024$.

321 Example $\log_3 27 = 3$ since $3^3 = 27$.

322 Example $\log_{190123456} 1 = 0$ as $190123456^0 = 1$.

> *If $a > 0, a \neq 1$, it should be clear that $\log_a 1 = 0,\ \log_a a = 1$, and in general $\log_a a^t = t$, where t is any real number.*

323 Example $\log_{\sqrt{2}} 8 = \log_{2^{1/2}} (2^{1/2})^6 = 6$.

324 Example $\log_{\sqrt{2}} 32 = \log_{2^{1/2}} (2^{1/2})^{10} = 10$.

325 Example $\log_{3\sqrt{3}} 81\sqrt[8]{27} = \log_{3^{3/2}} (3^{3/2})^{(2/3)(35/8)} = \dfrac{2}{3} \cdot \dfrac{35}{8} = \dfrac{35}{12}$.

Aliter: We seek a solution x to

$$(3\sqrt{3})^x = 81\sqrt[8]{27}$$

[1] In higher mathematics, and in many computer algebra programmes like Maple ®, the notation "log" without indicating the base, is used for the natural logarithm of base e. Misguided authors, enemies of the State, communists,Al-Qaeda members, vegetarians and other vile criminals use "log" in calculators and in lower mathematics to denote the logarithm of base 10, and use "ln" to denote the natural logarithm. This makes things somewhat confusing. In these notes we will denote the logarithm base 10 by "$\log_{10}$" and the natural logarithm by "$\log_e$", which is hardly original but avoids confusion.

Expressing the sinistral side as powers of 3, we have

$$\begin{aligned}(3\sqrt{3})^x &= (3\cdot 3^{1/2})^x\\ &= (3^{1+1/2})^x\\ &= (3^{3/2})^x\\ &= 3^{3x/2}\end{aligned}$$

Also, the dextral side equals

$$\begin{aligned}81\sqrt[8]{27} &= 3^4\cdot(3^3)^{1/8}\\ &= 3^{4+3/8}\\ &= 3^{35/8}\end{aligned}$$

Thus $(3\sqrt{3})^x = 81\sqrt[8]{27}$ implies that $3^{3x/2} = 3^{35/8}$ or $\dfrac{3x}{2} = \dfrac{35}{8}$ from where $x = \frac{35}{12}$.

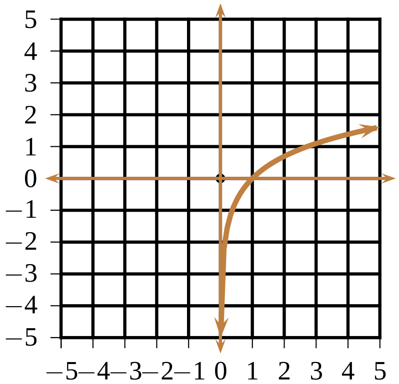

Figure 8.1: $x \mapsto \log_a x, a > 1$

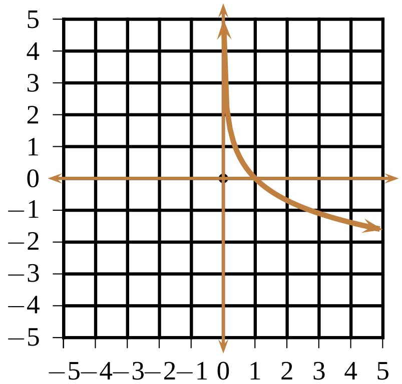

Figure 8.2: $x \mapsto \log_a x, 0 < a < 1$.

Since $x \mapsto a^x$ and $x \mapsto \log_a x$ are inverses, the graph of $x \mapsto \log_a x$ is symmetric with respect to the line $y = x$ to the graph of $x \mapsto a^x$. For $a > 1, x \mapsto a^x$ is increasing and convex, $x \mapsto \log_a x,\ a > 1$ will be increasing and concave, as in figure 8.1. Also, for $0 < a < 1, x \mapsto a^x$ is decreasing and convex, $x \mapsto \log_a x,\ 0 < a < 1$ will be decreasing and concave, as in figure 8.2.

326 Example Between which two integers does $\log_2 1000$ lie?

> **Solution:** ► *Observe that* $2^9 = 512 < 1000 < 1024 = 2^{10}$. *Since* $x \mapsto \log_2 x$ *is increasing, we deduce that* $\log_2 1000$ *lies between* 9 *and* 10. ◄

327 Example Find $\lfloor \log_3 201 \rfloor$.

> **Solution:** ► $3^4 = 81 < 201 < 243 = 3^5$. *Hence* $\lfloor \log_3 201 \rfloor = 4$. ◄

328 Example Which is greater $\log_5 7$ or $\log_8 3$?

> **Solution:** ► *Clearly* $\log_5 7 > 1 > \log_8 3$. ◄

329 Example Find the integer that equals

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \lfloor \log_2 4 \rfloor + \cdots + \lfloor \log_2 66 \rfloor.$$

Solution: ► *Firstly,* $\log_2 1 = 0$. *We may decompose the interval* $[2;66]$ *into dyadic blocks, as*

$$[2;66] = [2;4[\cup[4;8[\cup[8;16[\cup[16,;32[\cup[32,;64[\cup[64;66].$$

On the first interval there are $4-2=2$ *integers with* $\lfloor \log_2 x \rfloor = 1, x \in [2;4[$. *On the second interval there are* $8-4=4$ *integers with* $\lfloor \log_2 x \rfloor = 2, x \in [4;8[$. *On the third interval there are* $16-8=8$ *integers with* $\lfloor \log_2 x \rfloor = 3, x \in [8;16[$. *On the fourth interval there are* $32-16=16$ *integers with* $\lfloor \log_2 x \rfloor = 4, x \in [16;32[$. *On the fifth interval there are* $64-32=32$ *integers with* $\lfloor \log_2 x \rfloor = 5, x \in [32;64[$. *On the sixth interval there are* $66-64+1=3$ *integers with* $\lfloor \log_2 x \rfloor = 6, x \in [64;66]$. *Thus*

$$\begin{aligned}
\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + & \\
+ \lfloor \log_2 4 \rfloor + \cdots + \lfloor \log_2 66 \rfloor & \\
&= 2(1) + 4(2) + 8(3) + \\
&\quad +16(4) + 32(5) + 3(6) \\
&= 276.
\end{aligned}$$

◄

330 Example What is the natural domain of definition of $x \mapsto \log_2(x^2 - 3x - 4)$?

Solution: ► *We need* $x^2 - 3x - 4 = (x-4)(x+1) > 0$. *By making a sign diagram, or looking at the graph of the parabola* $y = (x-4)(x+1)$ *we see that this occurs when* $x \in]-\infty;-1[\cup]4;+\infty[$. ◄

331 Example What is the natural domain of definition of $x \mapsto \log_{|x|-4}(2-x)$?

Solution: ► *We need* $2-x>0$ *and* $|x|-4 \neq 1$. *Thus* $x<2$ *and* $x \neq 5, x \neq -5$. *We must have* $x \in]-\infty;-5[\cup]-5;2[$. ◄

Homework

8.1.1 Problem True or False.

1. $\exists x \in \mathbb{R}$ such that $\log_4 x = 2$.
2. $\exists x \in \mathbb{R}$ such that $\log_4 x = -2$.
3. $\log_2 1 = 0$.
4. $\log_2 0 = 1$.
5. $\log_2 2 = 1$.
6. $x \mapsto \log_{1/5} x$ is increasing over $\mathbb{R}_+^*$.
7. $\forall x > 0, (\log_5 x)^2 = \log_5 x^2$.
8. $\log_3 201 = 4$.

8.1.2 Problem Compute the following.

1. $\log_{1/3} 243$
2. $\log_{10} .00001$
3. $\log_{.001} 100000$
4. $\log_9 \dfrac{1}{3}$
5. $\log_{1024} 64$
6. $\log_{5^{2/3}} 625$
7. $\log_{2\sqrt{2}} 32\sqrt[5]{2}$
8. $\log_2 .0625$
9. $\log_{.0625} 2$
10. $\log_3 \sqrt[4]{729\sqrt[3]{9^{-1}27^{-4/3}}}$

8.1.3 Problem Let $a > 0, a \neq 1$. Compute the following.

1. $\log_a \sqrt[4]{a^{8/5}}$
2. $\log_a \sqrt[3]{a^{-15/2}}$
3. $\log_a \dfrac{1}{a^{1/2}}$
4. $\log_{a^3} a^6$
5. $\log_{a^2} a^3$
6. $\log_{a^{5/6}} a^{7/25}$

8.1.4 Problem Make a rough sketch of the following.

1. $x \mapsto \log_2 x$
2. $x \mapsto \log_2 |x|$
3. $x \mapsto 4 + \log_{1/2} x$
4. $x \mapsto 5 - \log_3 x$
5. $x \mapsto 2 - \log_{1/4} x$

6. $x \mapsto \log_5 x$
7. $x \mapsto \log_5 |x|$
8. $x \mapsto |\log_5 x|$
9. $x \mapsto |\log_5 |x||$
10. $x \mapsto 2 + \log_e |x|$
11. $x \mapsto -3 + \log_{1/2} |x|$
12. $x \mapsto 5 - |\log_4 x|$

8.1.5 Problem Prove that for $x > 0$,

$$1 - x \leq -\log_e x.$$

8.1.6 Problem Prove that for $x > 0$ we have

$$x^e \leq e^x.$$

Use this to prove that for $x > 0$,

$$\log_e x \leq \frac{x}{e}.$$

8.1.7 Problem Find the natural domain of definition of the following.

1. $x \mapsto \log_2(x^2 - 4)$
2. $x \mapsto \log_2(x^2 + 4)$
3. $x \mapsto \log_2(4 - x^2)$
4. $x \mapsto \log_2(\frac{x+1}{x-2})$
5. $x \mapsto \log_{x^2+1}(x^2 + 1)$
6. $x \mapsto \log_{1-x^2} x$

8.2 Simple Exponential and Logarithmic Equations

Recall that for $a > 0,\ a \neq 1,\ b > 0$ the relation $a^x = b$ entails $x = \log_a b$. This proves useful in solving the following equations.

332 Example Solve the equation

$$\log_4 x = -3.$$

Solution: ► $x = 4^{-3} = \dfrac{1}{64}$. ◄

333 Example Solve the equation

$$\log_2 x = 5.$$

Solution: ► $x = 2^5 = 32$. ◄

334 Example Solve the equation

$$\log_x 16 = 2.$$

Solution: ► $16 = x^2$. *Since the base must be positive, we have* $x = 4$. ◄

335 Example Solve the equation $3^x = 2$.

Solution: ► *By definition,* $x = \log_3 2$. ◄

336 Example Solve the equation $9^x - 5 \cdot 3^x + 6 = 0$.

Solution: ► *We have*

$$9^x - 5 \cdot 3^x + 6 = (3^x)^2 - 5 \cdot 3^x + 6 = (3^x - 2)(3^x - 3).$$

Thus either $3^x - 2 = 0$ *or* $3^x - 3 = 0$. *This implies that* $x = \log_3 2$ *or* $x = 1$. ◄

337 Example Solve the equation $25^x - 5^x - 6 = 0$.

Solution: ► *We have*

$$25^x - 5^x - 6 = (5^x)^2 - 5^x - 6 = (5^x + 2)(5^x - 3),$$

whence $5^x - 3 = 0$ *or* $x = \log_5 3$ *as* $5^x + 2 = 0$ *does not have a real solution. (Why?)* ◄

Since $x \mapsto a^x$ and $x \mapsto \log_a x$ are inverses, we have

$$x = a^{\log_a x}\ \forall a > 0,\ a \neq 1,\ \forall x > 0 \tag{8.1}$$

Thus for example, $5^{\log_5 4} = 4,\ 26^{\log_{26} 8} = 8$. This relation will prove useful in solving some simple equations.

338 Example Solve the equation

$$\log_2 \log_4 x = -1.$$

Solution: ► *As* $\log_2 \log_4 x = -1$, *we have*

$$\log_4 x = 2^{\log_2 \log_4 x} = 2^{-1} = \frac{1}{2}.$$

Hence $x = 4^{\log_4 x} = 4^{1/2} = \sqrt{4} = 2$. ◄

339 Example Solve the equation

$$\log_2 \log_3 \log_5 x = 0$$

Solution: ► *Since* $\log_2 \log_3 \log_5 x = 0$ *we have*

$$\log_3 \log_5 x = 2^{\log_2 \log_3 \log_5 x} = 2^0 = 1.$$

Hence

$$\log_5 x = 3^{\log_3 \log_5 x} = 3^1 = 3.$$

Finally $x = 5^{\log_5 x} = 5^3 = 125$. ◄

340 Example Solve the equation

$$\log_2 x(x-1) = 1.$$

Solution: ► *We have* $x(x-1) = 2^1 = 2$. *Hence* $x^2 - x - 2 = 0$. *This gives* $x = 2$ *or* $x = -1$. *Check that both are indeed solutions!* ◄

341 Example Solve the equation $\log_{e+x} e^8 = 2$.

Solution: ► *We have* $(e+x)^2 = e^8$ *or* $e + x = \pm e^4$. *Now the base* $e + x$ *cannot be negative, so we discard the minus sign alternative. The only solution is when* $e + x = e^4$, *that is,* $x = e^4 - e$. ◄

Homework

8.2.1 Problem Find real solutions to the following equations for x.

1. $\log_x 3 = 4$
2. $\log_3 x = 4$
3. $\log_4 x = 3$
4. $\log_{x-2} 9 = 2$
5. $\log_{|x|} 16 = 4$
6. $23^x - 2 = 0$
7. $(2^x - 3)(3^x - 2)(6^x - 1) = 0$
8. $4^x - 9 \cdot 2^x + 14 = 0$
9. $49^x - 2 \cdot 7^x + 1 = 0$
10. $36^x - 2 \cdot 6^x = 0$
11. $36^x + 6^x - 6 = 0$
12. $5^x + 12 \cdot 5^{-x} = 7$
13. $\log_2 \log_3 x = 2$
14. $\log_3 \log_5 x = -1$

8.3 Properties of Logarithms

A few properties of logarithms that will simplify operations with them will now be deduced.

342 Theorem If $a > 0, a \neq 1,\ M > 0$, and α is any real number, then

$$\log_a\ M^{\alpha} = \alpha \log_a\ M \tag{8.2}$$

Proof: *Let $x = \log_a\ M$. Then $a^x = M$. Raising both sides of this equality to the exponent α, one gathers $a^{\alpha x} = M^{\alpha}$. But this entails that $\log_a M^{\alpha} = \alpha x = \alpha(\log_a\ M)$, which proves the theorem.* ❑

343 Example How many digits does 8^{330} have?

Solution: ► *Let n be the integer such that $10^n < 8^{330} < 10^{n+1}$. Clearly then 8^{330} has $n+1$ digits. Since $x \mapsto \log_{10} x$ is increasing, taking logarithms base 10 one has $n < 330\log_{10} 8 < n+1$. Using a calculator, we see that $298.001 < 330\log_{10} 8 < 298.02$, whence $n = 298$ and so 8^{330} has 299 digits.* ◄

344 Example If $\log_a t = 2$, then $\log_a t^3 = 3\log_a t = 3(2) = 6$.

345 Example $\log_5\ 125 = \log_5\ 5^3 = 3\log_5\ 5 = 3(1) = 3$.

346 Theorem Let $a > 0, a \neq 1,\ M > 0$, and let $\beta \neq 0$ be a real number. Then

$$\log_{a^{\beta}}\ M = \frac{1}{\beta}\log_a\ M. \tag{8.3}$$

Proof: *Let $x = \log_a\ M$. Then $a^x = M$. Raising both sides of this equality to the power $\frac{1}{\beta}$ we gather $a^{x/\beta} = M^{1/\beta}$. But this entails that*

$$\log_a M^{1/\beta} = \frac{x}{\beta} = \frac{1}{\beta}(\log_a\ M),$$

which proves the theorem. ❑

347 Example Given that $\log_{8\sqrt{2}} 1024$ is a rational number, find it.

Solution: ► *We have*

$$\log_{8\sqrt{2}} 1024 = \log_{2^{7/2}} 1024 = \frac{2}{7}\log_2 2^{10} = \frac{2}{7}\cdot 10\log_2\ 2 = \frac{20}{7}.$$

◄

348 Theorem If $a > 0, a \neq 1,\ M > 0,\ N > 0$ then

$$\log_a\ MN = \log_a\ M + \log_a\ N \tag{8.4}$$

In words, the logarithm of a product is the sum of the logarithms.

Proof: *Let $x = \log_a\ M$ and let $y = \log_a\ N$. Then $a^x = M$ and $a^y = N$. This entails that $a^{x+y} = a^x a^y = MN$. But $a^{x+y} = MN$ entails $x + y = \log_a\ MN$, that is*

$$\log_a\ M + \log_a\ N = x + y = \log_a\ MN,$$

as required. ❑

349 Example If $\log_a\ t = 2,\ \log_a\ p = 3$ and $\log_a\ u^3 = 21$, find $\log_a\ t^3 pu$.

Solution: ► *First observe that $\log_a\ t^3 pu = \log_a\ t^3 + \log_a\ p + \log_a u$. Now, $\log_a t^3 = 3\log_a\ t = 6$. Also, $21 = \log_a u^3 = 3\log_a\ u$, from where $\log_a\ u = 7$. Hence*

$$\log_a\ t^3 pu = \log_a\ t^3 + \log_a\ p + \log_a u = 6 + 3 + 7 = 16.$$

◄

350 Example Solve the equation

$$\log_2 x + \log_2(x-1) = 1.$$

Solution: ► *If* $x > 1$ *then*

$$\log_2 x + \log_2(x-1) = \log_2 x(x-1).$$

This entails $x(x-1) = 2$, *from where* $x = -1$ *or* $x = 2$. *The solution* $x = -1$ *must be discarded, as we need* $x > 1$. ◄

351 Theorem If $a > 0, a \neq 1,\ M > 0,\ N > 0$ then

$$\log_a \frac{M}{N} = \log_a M - \log_a N \tag{8.5}$$

Proof: *Let* $x = \log_a M$ *and let* $y = \log_a N$. *Then* $a^x = M$ *and* $a^y = N$. *This entails that* $a^{x-y} = \dfrac{a^x}{a^y} = \dfrac{M}{N}$. *But* $a^{x-y} = \dfrac{M}{N}$ *entails* $x - y = \log_a \frac{M}{N}$, *that is*

$$\log_a M - \log_a N = x - y = \log_a \frac{M}{N},$$

as required. ❑

352 Example Let $\log_a t = 2,\ \log_a p = 3$ and $\log_a u^3 = 21$, find $\log_a \dfrac{p^2}{tu}$.

Solution: ► *First observe that*

$$\log_a \frac{p^2}{tu} = \log_a p^2 - \log_a tu = 2\log_a p - (\log_a t + \log_a u).$$

This entails that

$$\log_a \frac{p^2}{tu} = 2(3) - (2+21) = -17.$$

◄

353 Theorem If $a > 0, a \neq 1,\ b > 0, b \neq 1$ and $M > 0$ then

$$\log_a M = \frac{\log_b M}{\log_b a}. \tag{8.6}$$

Proof: *From the identity* $b^{\log_b M} = M$, *we obtain, upon taking logarithms base a on both sides*

$$\log_a \left(b^{\log_b M}\right) = \log_a M.$$

By Theorem 3.4.1

$$\log_a \left(b^{\log_b M}\right) = (\log_b M)(\log_a b),$$

whence the theorem follows. ❑

354 Example Given that

$$(\log_2 3) \cdot (\log_3 4) \cdot (\log_4 5) \cdots (\log_{511} 512)$$

is an integer, find it.

Solution: ► *Choose* $a > 0, a \neq 1$. *Then*

$$\begin{aligned}(\log_2 3)\cdot(\log_3 4)\cdot(\log_4 5)\cdots(\log_{511} 512) &= \frac{\log_a 3}{\log_a 2}\cdot\frac{\log_a 4}{\log_a 3}\cdot\frac{\log_a 5}{\log_a 4}\cdots\frac{\log_a 512}{\log_a 511}\\ &= \frac{\log_a 512}{\log_a 2}.\end{aligned}$$

But

$$\frac{\log_a 512}{\log_a 2} = \log_2 512 = \log_2 2^9 = 9,$$

so the integer sought is 9. ◄

355 Corollary If $a > 0, a \neq 1,\ b > 0, b \neq 1$ then

$$\log_a\ b = \frac{1}{\log_b\ a}. \tag{8.7}$$

Proof: *Let* $M = b$ *in the preceding theorem.* ❑

356 Example Given that $\log_n\ t = 2a,\ \log_s n = 3a^2$, find $\log_t s$ in terms of a.

Solution: ► *We have*

$$\log_t s = \frac{\log_n s}{\log_n t}.$$

Now, $\log_n s = \dfrac{1}{\log_s n} = \dfrac{1}{3a^2}$. *Hence*

$$\log_t s = \frac{\log_n s}{\log_n t} = \frac{\frac{1}{3a^2}}{2a} = \frac{1}{6a^3}.$$

◄

357 Example Given that $\log_a 3 = s^{-3},\ \log_{\sqrt{3}} b = s^2 + 2,\ \log_9 c = s^3$, write $\log_3 \dfrac{a^2b^5}{c^4}$ as a polynomial in s.

Solution: ► *Observe that*

$$\log_3 \frac{a^2b^5}{c^4} = 2\log_3 a + 5\log_3 b - 4\log_3 c,$$

so we seek information about $\log_3 a, \log_3 b$ *and* $\log_3 c$. *Now,*

$$\log_3 a = \frac{1}{\log_a 3} = s^3, \quad \log_3 b = \frac{1}{2}\log_{\sqrt{3}} b = \frac{1}{2}s^2 + 1$$

and $\log_3 c = 2\log_9 c = 2s^3$. *Hence*

$$\log_3 \frac{a^2b^5}{c^4} = 2s^3 + \frac{5}{2}s^2 + 5 - 8s^3 = -6s^3 + \frac{5}{2}s^2 + 5.$$

◄

358 Example Given that $.63 < \log_3\ 2 < .631$, find the smallest positive integer a such that $3^a > 2^{102}$.

Solution: ► *Since* $x \mapsto \log_3\ x$ *is an increasing function, we have* $a\log_3\ 3 > 102\log_3\ 2$*, that is,* $a > 102\log_3\ 2$. *Using the given information,* $64.26 < 102\log_3\ 2 < 64.362$*, which means that* $a = 65$ *is the smallest such integer.*
◄

359 Example Assume that there is a positive real number x such that

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2,$$

where there is an infinite number of x's. What is the value of x?

Solution: ► *Since* $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$*, one has*

$$2 = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x^2,$$

whence, as x is positive, $x = \sqrt{2}$. ◄

Euler shewed that the equation

$$a^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x$$

has real roots only for $a \in [e^{-e}; e^{1/e}]$.

360 Example How many real positive solutions does the equation

$$x^{(x^x)} = (x^x)^x$$

have?

Solution: ► *Assuming* $x > 0$ *we have* $x^x \log_e x = x \log_e x^x$ *or* $x^x \log_e x = x^2 \log_e x$*. Thus* $(\log_e x)(x^x - x^2) = 0$*. Thus either* $\log_e x = 0$*, in which case* $x = 1$*, or* $x^x = x^2$*, in which case* $x = 2$*. The equation has therefore only two positive solutions.* ◄

361 Example The non-negative integers smaller than 10^n are split into two subsets A and B. The subset A contains all those integers whose decimal expansion does not contain a 5, and the set B contains all those integers whose decimal expansion contains at least one 5. Given n, which subset, A or B is the larger set? One may use the fact that $\log_{10} 2 := .3010$ and that $\log_{10} 3 := .4771$.

Solution: ► *The set B contains* $10^n - 9^n$ *elements and the set A contains* 9^n *elements. Now if* $10^n - 9^n > 9^n$ *then* $10^n > 2 \cdot 9^n$ *and taking logarithms base 10 we deduce*

$$n > \log_{10} 2 + 2n \log_{10} 3.$$

Thus

$$n > \frac{\log_{10} 2}{1 - 2\log_{10} 3} := 6.57\ldots$$

Therefore, if $n \leq 6$*, A has more elements than B and if* $n > 6$*, B has more elements than A.* ◄

362 Example Shew that if $a, b, c,$ are real numbers with $a^2 = b^2 + c^2, a+b > 0, a+b \neq 1, a-b > 0, a-b \neq 1$, then

$$\log_{a-b} c + \log_{a+b} c = 2(\log_{a-b} c)(\log_{a+b} c).$$

Solution: ► *As* $c^2 = a^2 - b^2 = (a-b)(a+b)$*, upon taking logarithms base* $a+b$ *we have*

$$2\log_{a+b} c = \log_{a+b} (a-b)(a+b) = 1 + \log_{a+b} (a-b) \tag{8.8}$$

Similarly, taking logarithms base $a-b$ *on the identity* $c^2 = (a-b)(a+b)$ *we obtain*

$$2\log_{a-b} c = \log_{a-b} (a-b)(a+b) = 1 + \log_{a-b} (a+b) \tag{8.9}$$

Multiplying these last two identities,

$$\begin{aligned}
4(\log_{a-b}\, c)(\log_{a+b}\, c) &= (1+\log_{a+b}\,(a-b))(1+\log_{a-b}\,(a+b)) \\
&= 1+\log_{a-b}\,(a+b)+\log_{a+b}\,(a-b) \\
&\quad +(\log_{a-b}\,(a+b))(\log_{a+b}\,(a-b)) \\
&= 2+\log_{a-b}\,(a+b)+\log_{a+b}\,(a-b) \\
&= 2+\log_{a-b}\,\tfrac{c}{a-b}+\log_{a+b}\,\tfrac{c}{a+b} \\
&= \log_{a-b}\, c+\log_{a+b}\, c,
\end{aligned}$$

as we wanted to shew. ◀

363 Example If $\log_{12}\, 27 = a$ prove that $\log_6\, 16 = \dfrac{4(3-a)}{3+a}$.

Solution: ► *First notice that* $a = \log_{12}\, 27 = 3\log_{12}\, 3 = \dfrac{3}{\log_3\, 12} = \dfrac{3}{1+2\log_3\, 2}$, *whence* $\log_3\, 2 = \dfrac{3-a}{2a}$ *or* $\log_2\, 3 = \dfrac{2a}{3-a}$. *Also*

$$\begin{aligned}
\log_6\, 16 &= 4\log_6\, 2 \\
&= \tfrac{4}{\log_2\, 6} \\
&= \tfrac{4}{1+\log_2\, 3} \\
&= \tfrac{4}{1+\frac{2a}{3-a}} \\
&= \tfrac{4(3-a)}{3+a},
\end{aligned}$$

as required. ◀

364 Example Solve the system

$$5\left(\log_x y+\log_y x\right) = 26$$
$$xy = 64$$

Solution: ► *Clearly we need* $x>0, y>0, x\neq 1, y\neq 1$. *The first equation may be written as* $5\left(\log_x y+\dfrac{1}{\log_x y}\right) = 26$ *which is the same as* $(\log_x y-5)(\log_y x-\dfrac{1}{5}) = 0$. *Thus the system splits into the two equivalent systems (I)* $\log_x y = 5, xy = 64$ *and (II)* $\log_x y = 1/5, xy = 64$. *Using the conditions* $x>0, y>0, x\neq 1, y\neq 1$ *we obtain the two sets of solutions* $x = 2, y = 32$ *or* $x = 32, y = 2$. ◀

Homework

8.3.1 Problem Find the exact value of

$$\frac{1}{\log_2 1996!}+\frac{1}{\log_3 1996!}+\frac{1}{\log_4 1996!}+\cdots+\frac{1}{\log_{1996} 1996!}.$$

8.3.2 Problem

1. $\log_4 MN = \log_4 M+\log N\ \forall M,N\in\mathbb{R}$.
2. $\log_5 M^2 = 2\log_5 M \forall M\in\mathbb{R}$.
3. $\exists\, M\in\mathbb{R}$ such that $\log_5 M^2 = 2\log_5 M$.

8.3.3 Problem Given that $\log_a p = 2,\ \log_a m = 9,\ \log_a n = -1$ find

1. $\log_a p^7$
2. $\log_{a^7} p$
3. $\log_{a^4} p^2 n^3$

4. $\log_{a^6} \dfrac{m^3 n}{p^6}$

8.3.4 Problem Which number is larger, 3^{1000} or 5^{600}?

8.3.5 Problem Find $(\log_3 169)(\log_{13} 243)$ without recourse of a calculator or tables.

8.3.6 Problem Find $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$ without recourse of a calculator or tables.

8.3.7 Problem Given that $\log_a p = b,\ \log_q a = 3b^{-2}$, find $\log_p q$ in terms of b.

8.3.8 Problem Given that $\log_2 a = s,\ \log_4 b = s^2,\ \log_{c^2} 8 = \frac{2}{s^3+1}$, write $\log_2 \dfrac{a^2 b^5}{c^4}$ as a function of s.

8.3.9 Problem Given that $\log_{a^2}(a^2+1) = 16$, find the value of

$$\log_{a^{32}} \left(a + \tfrac{1}{a}\right).$$

8.3.10 Problem Write without logarithms. Assume the proper restrictions on the variables wherever necessary.

1. $(a^\alpha)^{-\beta \log_{\alpha S} N^\gamma}$
2. $-\log_8 \log_4 \log_2 16$
3. $\log_{0.75} \log_2 \sqrt{\sqrt[-2]{0.125}}$
4. $\left(5^{(\log_7 5)^{-1}} + (-\log_{10} 0.1)^{-1/2}\right)^{1/3}$
5. $b^{a^{(\log_b \log_b N)/(\log_b a)}}$
6. $2^{(\log_3 5)} - 5^{(\log_3 2)}$
7. $\left(\dfrac{1}{49}\right)^{1+(\log_7 2)} + 5^{-(\log_{1/5} 7)}$

8.3.11 Problem A sheet of paper has approximately 0.1 mm of thickness. Suppose you fold the sheet by halves, thirty times consecutively. (1) What is the thickness of the folded paper?, (2) How many times should you fold the sheet in order to obtain the distance from Earth to the Moon? (the distance from Earth to the Moon is about 384 000 km.)

8.3.12 Problem How many digits does 11^{2000} have?

8.3.13 Problem Let $A = \log_6 16, B = \log_{12} 27$. Find integers a, b, c such that $(A+a)(B+b) = c$.

8.3.14 Problem Given that $\log_{ab} a = 4$, find

$$\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}}.$$

8.3.15 Problem The number 5^{100} is written in binary (base-2) notation. How many binary digits does it have?

8.3.16 Problem Prove that if $x > 0, a > 0, a \neq 1$ then $x^{1/\log_a x} = a$.

8.3.17 Problem Let a, b, x be positive real numbers distinct from 1. When is it true that

$$4(\log_a x)^2 + 3(\log_b x)^2 = 8(\log_a x)(\log_b x)\ ?$$

8.3.18 Problem Prove that $\log_3 \pi + \log_\pi 3 > 2$.

8.3.19 Problem Solve the equation

$$4 \cdot 9^{x-1} = 3\sqrt{2^{2x+1}}$$

8.3.20 Problem Solve the equation

$$5^{x-1} + 5(0.2)^{x-2} = 26$$

8.3.21 Problem Solve the equation

$$25^x - 12 \cdot 2^x - (6.25)(0.16)^x = 0$$

8.3.22 Problem Solve the equation

$$\log_3(3^x - 8) = 2 - x$$

8.3.23 Problem Solve the equation

$$\log_4(x^2 - 6x + 7) = \log_4(x - 3)$$

8.3.24 Problem Solve the equation

$$\log_3(2-x) - \log_3(2+x) - \log_3 x + 1 = 0$$

8.3.25 Problem Solve the equation

$$2\log_4(2x) = \log_4(x^2 + 75)$$

8.3.26 Problem Solve the equation

$$\log_2(2x) = \frac{1}{4}\log_2(x-15)^4$$

8.3.27 Problem Solve the equation

$$\frac{\log_2 x}{\log_4 2x} = \frac{\log_8 4x}{\log_{16} 8x}$$

8.3.28 Problem Solve the equation

$$\log_3 x = 1 + \log_x 9$$

8.3.29 Problem Solve the equation

$$25^{\log_2 x} = 5 + 4x^{\log_2 5}$$

8.3.30 Problem Solve the equation

$$x^{\log_{10} 2x} = 5$$

8.3.31 Problem Solve the equation

$$|x-3|^{(x^2-8x+15)/(x-2)} = 1$$

8.3.32 Problem Solve the equation

$$\log_{2x-1}\frac{x^4+2}{2x+1} = 1$$

8.3.33 Problem Solve the equation

$$\log_{3x} x = \log_{9x} x$$

8.3.34 Problem Solve

$$\log_2 x + \log_4 y + \log_4 z = 2,$$

$$\log_3 x + \log_9 y + \log_9 z = 2,$$

$$\log_4 x + \log_{16} y + \log_{16} z = 2.$$

8.3.35 Problem Solve the equation

$$x^{0.5\log_{\sqrt{x}}(x^2-x)} = 3^{\log_9 4}.$$

9 Goniometric Functions

9.1 The Winding Function

Recall that a circle of radius r has a circumference of $2\pi r$ units of length. Hence a unit circle, i.e., one with $r = 1$, has circumference 2π.

365 Definition A *radian* is a $\dfrac{1}{2\pi}$th part of the circumference of a unit circle.

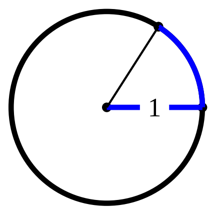

Figure 9.1: A radian.

Since $\frac{1}{2\pi} \approx 0.16$, a radian is about $\dfrac{4}{25}$ of the circumference of the unit circle. A quadrant or quarter part of a circle has arc length of $\frac{\pi}{4}$ radians. A semicircle has arc length $\frac{2\pi}{2} = \pi$ radians.

1. *A radian is simply a real number!*
2. *If a central angle of a unit circle cuts an arc of x radians, then the central angle measures x radians.*
3. *The sum of the internal angles of a triangle is π radians.*

Suppose now that we cut a unit circle into a “string” and use this string to mark intervals of length 2π on the real line. We put an endpoint 0, mark off intervals to the right of 0 with endpoints at $2\pi, 4\pi, 6\pi, \ldots,$ etc. We start again, this time going to the left and marking off intervals with endpoints at $-2\pi, -4\pi, -6\pi, \ldots,$ etc., as shewn in figure 9.2.

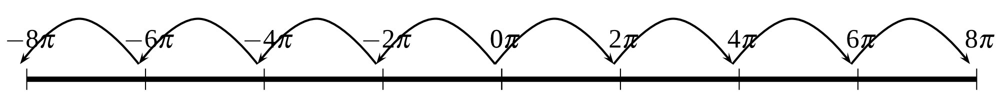

Figure 9.2: The Real Line modulo 2π.

We have decomposed the real line into the union of disjoint intervals

$$\ldots \cup [-6\pi; -4\pi[\cup [-4\pi; -2\pi[\cup [-2\pi; 0[\cup [0; 2\pi[\cup [2\pi; 4\pi[\cup [4\pi; 6\pi[\cup \ldots$$

Observe that each real number belongs to one, and only one of these intervals, that is, there is a unique integer k such that if $x \in \mathbb{R}$ then $x \in [2\pi k; (2k+2)\pi[$. For example $100 \in [30\pi; 32\pi[$ and $-9 \in [-4\pi; -2\pi[$.

366 Definition Given two real numbers a and b, we say that *a is congruent to b* modulo 2π, written $a \equiv b \mod 2\pi$, if $\dfrac{a-b}{2\pi}$ is an integer. If $\dfrac{a-b}{2\pi}$ is not an integer, we say that a and b are *incongruent* modulo 2π and we write $a \not\equiv b \mod 2\pi$.

For example, $5\pi \equiv -7\pi \mod 2\pi$, since $\dfrac{5\pi-(-7\pi)}{2\pi} = \dfrac{12\pi}{2\pi} = 6$, an integer. However, $5\pi \not\equiv 2\pi \mod 2\pi$ as $\dfrac{5\pi-2\pi}{2\pi} = \dfrac{3\pi}{2\pi} = \dfrac{3}{2}$, which is not an integer.

367 Definition If $a \equiv b \mod 2\pi$, we say that a and b belong to the same *residue class* $\mod 2\pi$. We also say that a and b are *representatives* of the same residue class modulo 2π.

368 Theorem Given a real number a, all the numbers of the form $a+2\pi k,\ k \in \mathbb{Z}$ belong to the same residue class modulo 2π.

> **Proof:** *Take two numbers of this form, $a+2\pi k_1$ and $a+2\pi k_2$, say, with integers k_1, k_2. Then*
>
> $$\frac{(a+2\pi k_1)-(a+2\pi k_2)}{2\pi} = k_1 - k_2,$$
>
> *which being the difference of two integers is an integer. This shews that $a+2\pi k_1 \equiv a+2\pi k_2 \mod 2\pi$.* ❑

369 Example Take $x = \frac{\pi}{3}$. Then

$$\begin{aligned}
\tfrac{\pi}{3} &\equiv \tfrac{\pi}{3}+2\pi \equiv \tfrac{7\pi}{3} \mod 2\pi \\
&\equiv \tfrac{\pi}{3}-2\pi \equiv -\tfrac{5\pi}{3} \mod 2\pi \\
&\equiv \tfrac{\pi}{3}+4\pi \equiv \tfrac{13\pi}{3} \mod 2\pi \\
&\equiv \tfrac{\pi}{3}-4\pi \equiv -\tfrac{11\pi}{3} \mod 2\pi
\end{aligned}$$

Thus all of

$$\frac{\pi}{3}, \frac{7\pi}{3}, -\frac{5\pi}{3}, \frac{13\pi}{3}, -\frac{11\pi}{3}$$

belong to the same residue class $\mod 2\pi$.

> *If $a \equiv b \mod 2\pi$ then there exists an integer k such that $a = b+2\pi k$.*

Given a real number x, it is clear that there are infinitely many representatives of the class to which x belongs, as we can add any integral multiple of 2π to x and still lie in the same class. However, exactly one representative x_0 lies in the interval $[0, 2\pi[$, as we saw above. We call x_0 the *canonical* representative of the class (to which x belongs modulo 2π).

To find the canonical representative of the class of x, we simply look for the integer k such that $2k\pi \le x < (2k+2)\pi$. Then then $0 \le x - 2k\pi < 2\pi$ and so $x - 2\pi k$ is the canonical representative of the class of x.

370 Definition We will call the procedure of finding a canonical representative for the class of x, *reduction* modulo 2π.

371 Example Reduce $5\pi \mod 2\pi$.

> **Solution:** ► *Since $4\pi < 5\pi < 6\pi$, we have $5\pi \equiv 5\pi - 4\pi \equiv \pi \mod 2\pi$. Thus π is the canonical representative of the class to which 5π belongs, modulo 2π.*
>
> ◄

To speed up the computations, we may avail of the fact that $2\pi k \equiv 0 \mod 2\pi$, that is, any integral multiple of 2π is congruent to $0 \mod 2\pi$.

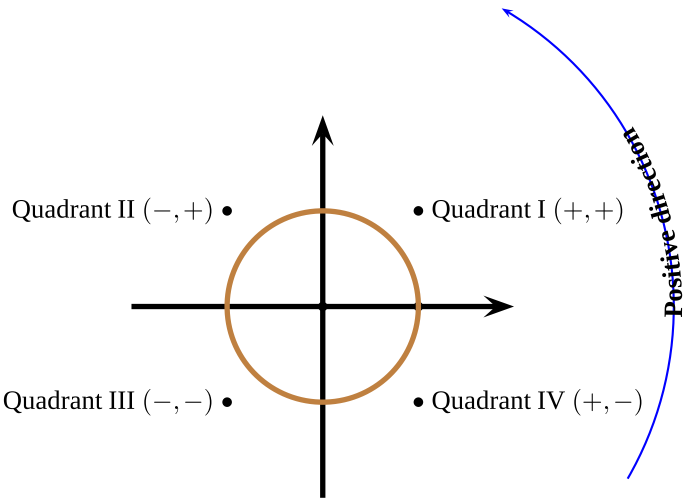

Figure 9.3: The unit circle on the Cartesian Plane.

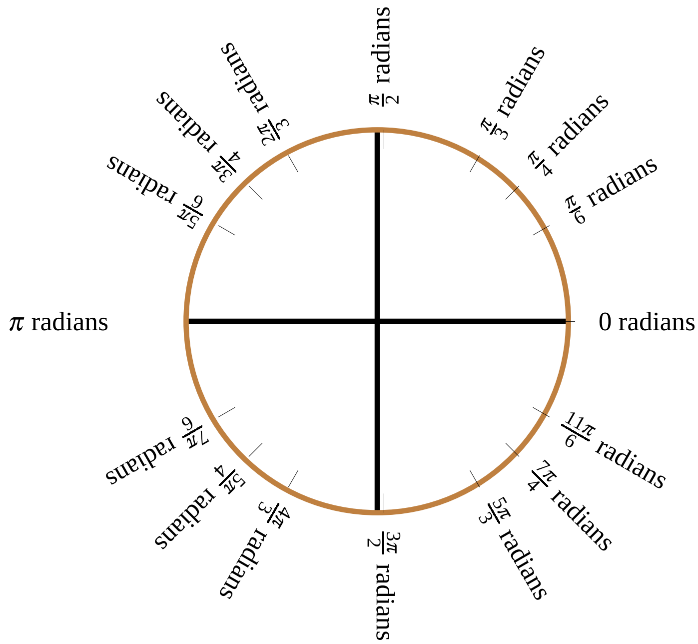

Figure 9.4: The unit circle on the Cartesian Plane.

372 Example Reduce $\dfrac{200\pi}{7}$ modulo 2π.

Solution: ▶ $\dfrac{200\pi}{7} \equiv \dfrac{196\pi + 4\pi}{7} \equiv 28\pi + \dfrac{4\pi}{7} \equiv \dfrac{4\pi}{7} \mod 2\pi$. ◀

373 Example Reduce $-\dfrac{5\pi}{7}$ modulo 2π.

Solution: ▶ $-\dfrac{5\pi}{7} \equiv 2\pi - \dfrac{5\pi}{7} \equiv \dfrac{9\pi}{7} \mod 2\pi$. ◀

374 Example Reduce $7 \mod 2\pi$.

Solution: ▶ *Since $2\pi < 6.29 < 7 < 4\pi$, the largest even multiple of π smaller than 7 is 2π, whence $7 \equiv 7 - 2\pi \mod 2\pi$.* ◀

Place now the centre of a unit circle at the origin of the Cartesian Plane. Choosing the point $(1,0)$ as our departing point (a completely arbitrary choice), we traverse the circumference of the unit circle counterclockwise (again, the choice is completely arbitrary). If we traverse 0 units, we are still at $(1,0)$, on the positive portion of the x-axis. If we traverse a number of units in the interval $\left]0; \frac{\pi}{2}\right[$, we are in the first quadrant.

If we have traversed exactly $\dfrac{\pi}{2}$ units, we are at $(0,1)$, on the positive portion of the y-axis. Traversing a number of units in the interval $\left]\frac{\pi}{2}; \pi\right[$, puts us in the second quadrant. If we travel exactly π units, we are at $(-1,0)$, the negative portion of the x-axis. Traversing a number of units in the interval $\left]\pi; \frac{3\pi}{2}\right[$, puts us in the third quadrant. Traversing exactly $\frac{3\pi}{2}$ units puts us at the point $(0,-1)$, the negative portion of the y-axis. Travelling a number of units in the interval $\left]\frac{3\pi}{2}; 2\pi\right[$, puts us in the fourth quadrant. Finally, travelling exactly 2π units brings us back to $(1,0)$. So, after one revolution around the unit circle, we are back in already travelled territory. See figure 9.3.

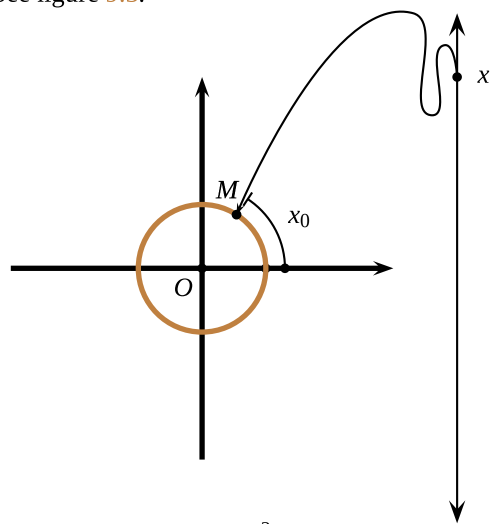

Figure 9.5: $\mathscr{C} : \mathbb{R} \to \mathbb{R}^2$, $\mathscr{C}(x) = M$.

If we traverse the unit circle clockwise, *then the arc length is measured negatively.*

We now define a function $\mathscr{C} : \mathbb{R} \to \mathbb{R}^2$ in the following fashion. Given a real number x, let x_0 be its canonical representative modulo 2π. Starting at $(1,0)$, traverse the circumference of the unit circle x_0 units counterclockwise. Your final destination is a point on the Cartesian Plane, call it M. We let $\mathscr{C}(x) = M$. See figure 9.5. The function $\mathscr{C}$ is called the *winding function*.

375 Example In what quadrant does $\mathscr{C}\left(-\frac{283\pi}{5}\right)$ lie?

Solution: ► *Observe that*

$$\begin{aligned} -\frac{283\pi}{5} &\equiv \frac{-280\pi-3\pi}{5} \\ &\equiv -56\pi - \frac{3\pi}{5} \\ &\equiv -\frac{3\pi}{5} \\ &\equiv 2\pi - \frac{3\pi}{5} \\ &\equiv \frac{7\pi}{5} \mod 2\pi. \end{aligned}$$

Since $\frac{7\pi}{5} \in]\pi; \frac{3\pi}{2}[$, $\mathscr{C}\left(-\frac{283\pi}{5}\right)$ *lies in quadrant III.* ◄

376 Example In what quadrant does $\mathscr{C}(451)$ lie?

Solution: ► *Since* $71 < \frac{451}{2\pi} < 71.8$, $142\pi < 451 < 144\pi$, *and hence* $451 \equiv 451 - 142\pi \mod 2\pi$. *Now,* $451 - 142\pi \approx 4.89 \in \left]\frac{3\pi}{2}; 2\pi\right[$, *and so* $\mathscr{C}(451)$ *lies in the fourth quadrant.* ◄

377 Example In which quadrant does $\mathscr{C}(\pi^2)$ lie?

Solution: ► *We multiply the inequality* $2 < \pi < 4$ *through by* π, *obtaining* $2\pi < \pi^2 < 4\pi$, *whence the largest even multiple of* π *less than* π^2 *is* 2π. *Therefore* $\pi^2 \equiv \pi^2 - 2\pi \mod 2\pi$. *Now we claim that*

$$\pi < \pi^2 - 2\pi < \frac{3\pi}{2}.$$

The sinistral inequality is easily deduced from the obvious inequality $3\pi < \pi^2$. The dextral inequality is deduced from the fact that $\pi^2 < 3.5\pi$. The inequality $\pi < \pi^2 - 2\pi < \frac{3\pi}{2}$ is thus proven, which means that $\mathscr{C}(\pi^2)$ lies in the third quadrant. ◀

378 Example Find the members of the set $\left\{\frac{\pi}{2} + \frac{k\pi}{3} : k \in \mathbb{Z}\right\}$ that belong to the interval $[8\pi; 10\pi[$.

Solution: ▶ *The problem is asking for all integers k such that*

$$8\pi \leq \frac{\pi}{2} + \frac{k\pi}{3} < 10\pi.$$

Now,

$$\begin{aligned} 8\pi \leq \tfrac{\pi}{2} + \tfrac{k\pi}{3} < 10\pi &\iff 8\pi - \tfrac{\pi}{2} \leq \tfrac{k\pi}{3} < 10\pi - \tfrac{\pi}{2} \\ &\iff \tfrac{15\pi}{2} \leq \tfrac{k\pi}{3} < \tfrac{19\pi}{2} \\ &\iff 22.5 \leq k < 28.5. \end{aligned}$$

Since k is an integer, $k \in \{23, 24, 25, 26, 27, 28\}$. The required elements are thus

$$\frac{\pi}{2} + \frac{23\pi}{3} = \frac{49\pi}{6}, \quad \frac{\pi}{2} + \frac{24\pi}{3} = \frac{17\pi}{2}, \quad \frac{\pi}{2} + \frac{25\pi}{3} = \frac{53\pi}{6},$$

$$\frac{\pi}{2} + \frac{26\pi}{3} = \frac{55\pi}{6}, \quad \frac{\pi}{2} + \frac{27\pi}{3} = \frac{19\pi}{2}, \quad \frac{\pi}{2} + \frac{28\pi}{3} = \frac{59\pi}{6}.$$

◀

379 Example Is $\dfrac{275\pi}{6} \in \left\{\frac{\pi}{2} + \frac{k\pi}{3} : k \in \mathbb{Z}\right\}$?

Solution: ▶ *The problem is asking whether there is an integer k such that*

$$\frac{275\pi}{6} = \frac{\pi}{2} + \frac{k\pi}{3}.$$

Solving for k we find $k = 136$, which is an integer. The answer is affirmative and indeed,

$$\frac{275\pi}{6} = \frac{\pi}{2} + \frac{136\pi}{3}.$$

◀

Homework

9.1.1 Problem True or False.

1. $10 \equiv 8 \mod 2\pi$.
2. $-\frac{9\pi}{7} \equiv \frac{5\pi}{7} \mod 2\pi$.
3. $\frac{1}{\pi} \equiv \frac{2}{\pi} \mod 2\pi$.
4. $\frac{7\pi}{6} \equiv \frac{\pi}{6} \mod 2\pi$.
5. $\frac{-8\pi}{41} \equiv -\frac{500\pi}{41} \mod 2\pi$.
6. $x \in [-1; 0[$ then $\mathscr{C}(x)$ is in quadrant IV.

9.1.2 Problem Reduce the following real numbers $\mod 2\pi$. Determine the quadrant in which their image under $\mathscr{C}$ would lie.

1. $\frac{3\pi}{5}$;
2. $-\frac{3\pi}{5}$;
3. $\frac{7\pi}{5}$,;
4. $\frac{8\pi}{57}$;
5. $\frac{57\pi}{8}$;
6. $\frac{6\pi}{79}$;
7. $\frac{790\pi}{7}$;
8. 1;
9. 2;
10. 3;
11. 4;
12. 5;
13. 6,;
14. 100,;
15. -3.14;
16. -3.15

9.1.3 Problem Find all the members of the set $\{\frac{3\pi}{4}+\frac{k\pi}{5} : k \in \mathbb{Z}\}$ that lie in the interval (i) $[0;\pi[$; (ii) $[-\pi;0[$.

9.1.4 Problem Is $\frac{279\pi}{20} \in \{\frac{3\pi}{4}+\frac{3k\pi}{5}|k\in\mathbb{Z}\}$? Is $-\frac{251\pi}{20} \in \{\frac{3\pi}{4}+\frac{3k\pi}{5}|k\in\mathbb{Z}\}$?

9.1.5 Problem Prove that congruence modulo 2π is reflexive, that is, if $a\in\mathbb{R}$, then $a\equiv a \mod 2\pi$.

9.1.6 Problem Prove that congruence modulo 2π is symmetric, that is, if $a,b\in\mathbb{R}$, and if $a\equiv b \mod 2\pi$ then $b\equiv a \mod 2\pi$.

9.1.7 Problem Prove that congruence modulo 2π is transitive, that is if if $a,b,c\in\mathbb{R}$, then $a\equiv b \mod 2\pi$ and $b\equiv c \mod 2\pi$ imply $a\equiv c \mod 2\pi$.

9.2 Cosines and Sines: Definitions

Consider any real number x. We find its canonical representative $x_0 \mod 2\pi$ and use this to find $\mathscr{C}(x)=M$, as in figure 9.6.

We now project the point M so obtained onto C and S on the axes. The *cosine* function $\begin{array}{ccc}\mathbb{R} & \to & [-1;1] \\ x & \mapsto & \cos x\end{array}$ is given by $\cos(x)=\cos x = OC$ (the algebraic length of the segment OC, that is, the signed distance from O to C) and the *sine* function $\begin{array}{ccc}\mathbb{R} & \to & [-1;1] \\ x & \mapsto & \sin x\end{array}$ is given by $\sin(x)=\sin x = OS$ (the algebraic length of the segment OS).

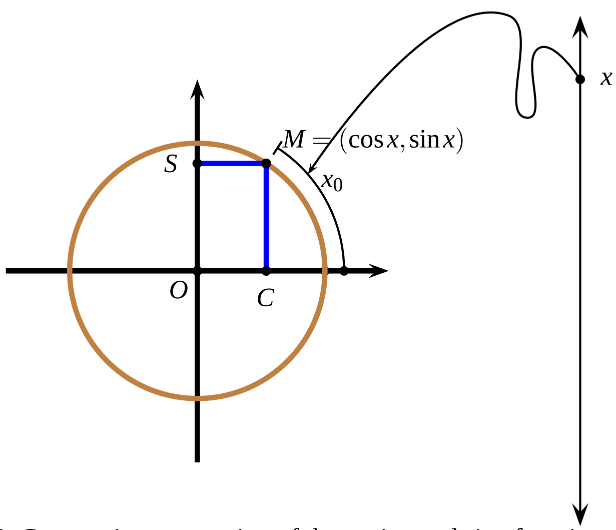

Figure 9.6: Geometric construction of the cosine and sine functions.

STOP

1. *The farthest right M can go is to $(1,0)$ and the farthest left is to $(-1,0)$. Thus $-1\le\cos x\le 1$. Similarly, the farthest up M can go is to $(0,1)$ and the farthest down it can go is to $(0,-1)$. Hence $-1\le\sin x\le 1$.*
2. *The sine and cosine functions are defined for all real numbers.*
3. *If $a\equiv b \mod 2\pi$ then $\cos a=\cos b$ and $\sin a=\sin b$. In other words, the cosine and sine functions are periodic with period 2π, that is*

$$\sin(2\pi+x)=\sin x \ \forall x\in\mathbb{R}, \qquad (9.1)$$

$$\cos(2\pi+x)=\cos x \ \forall x\in\mathbb{R}. \qquad (9.2)$$

4. *The point M has abscissa $\cos x$ and ordinate $\sin x$, that is, $M = (\cos x, \sin x)$.*

5. *The functions* $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \cos x \end{array}$ *and* $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \sin x \end{array}$ *are surjective (onto) but not injective (one-to-one).*

We may now compute some simple sines and cosines.

380 Example From figure 9.7, if $x = 0$ then the point M is $(1,0)$. Thus $\cos 0 = 1$, $\sin 0 = 0$. If $x = \dfrac{\pi}{2}$ the point M is $(0,1)$. From this we gather that $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$. If $x = \pi$ then the point M is $(-1,0)$. Thus $\cos\pi = -1$, $\sin\pi = 0$. If $x = \frac{3\pi}{2}$ the point M is $(0,1)$. From this we gather that $\cos\frac{3\pi}{2} = 0$ and $\sin\frac{3\pi}{2} = -1$.

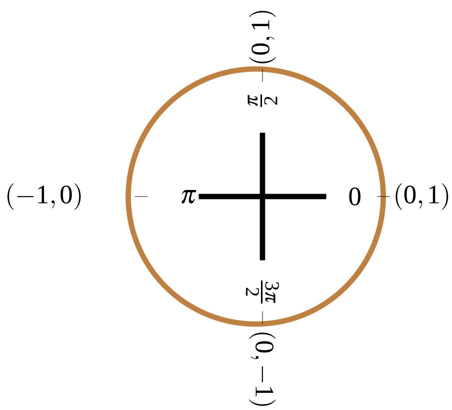

Figure 9.7: Some values of sin and cos.

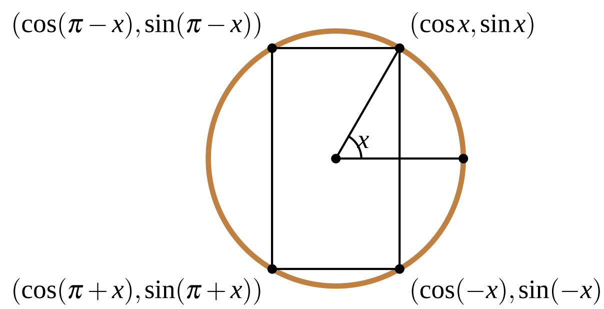

Figure 9.8: Symmetry Identities.

381 Definition If $K \neq -1$, we write $\sin^K x, \cos^K x$ to denote $(\sin x)^K, (\cos x)^K$, respectively. $\sin^{-1} x, \cos^{-1} x$, a are reserved for when we study inversion later in these notes.

The following relation, known as the *Pythagorean Relation* is fundamental in the study of circular functions.

382 Theorem (Pythagorean Relation) Let x be any real number. Then

$$\cos^2 x + \sin^2 x = 1. \tag{9.3}$$

Proof: *Let $\mathscr{C}(x) = M = (\cos x, \sin x)$, as in figure 9.6., where $O = (0,0)$, and S, C are the projections of M onto the axes. In $\triangle OCM$, $\cos x = OC$, and $\sin x = OS = CM$. As $\triangle OCM$ is a right triangle and $OM = 1$, by the Pythagorean Theorem, we have*

$$\cos^2 x + \sin^2 x = OC^2 + CM^2 = OM^2 = 1^2 = 1,$$

which completes the proof. ❑

STOP *Pay attention to the notation $\cos^2 x$ for $(\cos x)^2$ and respectively to $\sin^2 x$ for $(\sin x)^2$. Do not confuse these with $\cos x^2$ and $\sin x^2$. For example, if $x = \pi$ then $\cos^2\pi = (-1)^2 = 1$ and $\sin^2\pi = 0^2 = 0$. Since $\mathscr{C}(\pi^2)$ lies in the third quadrant, $\cos\pi^2 < 0$ and $\sin\pi^2 < 0$. Hence $\cos^2\pi \neq \cos\pi^2$ and $\sin^2\pi \neq \sin\pi^2$.*

From the Pythagorean Relation,

$$\cos x = \pm\sqrt{1 - \sin^2 x}$$

and

$$\sin x = \pm\sqrt{1-\cos^2 x}.$$

The ambiguity in sign is resolved by specifying in which quadrant $\mathscr{C}(x)$ lies, see figure 9.3.

383 Example Let $\frac{3\pi}{2} < x < 2\pi$ and $\cos x = \frac{1}{3}$. Find $\sin x$.

Solution: ▶ *$\mathscr{C}(x)$ lies in the fourth quadrant, where $\sin x < 0$. We have*

$$\sin x = -\sqrt{1-\cos^2 x} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}.$$

◀

384 Example Given that $\frac{\pi}{2} < x < \pi$, and that $\sin x = \frac{3}{5}$, find $\cos x$.

Solution: ▶ *Since $\mathscr{C}(x)$ lies in the second quadrant, the cosine is negative. Hence*

$$\cos x = -\sqrt{1-\sin^2 x} = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\frac{4}{5}.$$

◀

385 Theorem (Symmetry Identities) Let $x \in \mathbb{R}$. Then the following are identities.

$$\cos(-x) = \cos x, \tag{9.4}$$

$$\sin(-x) = -\sin x, \tag{9.5}$$

$$\cos(\pi - x) = -\cos x, \tag{9.6}$$

$$\sin(\pi - x) = \sin x, \tag{9.7}$$

$$\cos(\pi + x) = -\cos x, \tag{9.8}$$

$$\sin(\pi + x) = -\sin x, \tag{9.9}$$

Proof: *The first identity says that the cosine is an even function; the second that the sine is an odd function. The third and fourth identities are "supplementary angle" identities. The fifth and the sixth identities are a "reflexion about the origin." All of these identities can be derived at once from figure 9.8.* ❑

By the 2π-periodicity of the cosine and sine we have

$$\cos(2\pi k + x) = \cos x,\ \forall x \in \mathbb{R}\ \forall k \in \mathbb{Z} \tag{9.10}$$

$$\sin(2\pi k + x) = \sin x,\ \forall x \in \mathbb{R}\ \forall k \in \mathbb{Z}. \tag{9.11}$$

Now,

$$\cos((2k+1)\pi + x) = \cos(2\pi k + \pi + x) = \cos(\pi + x) = -\cos x$$

and

$$\sin((2k+1)\pi + x) = \sin(2\pi k + \pi + x) = \sin(\pi + x) = -\sin x,$$

whence the following corollary is proved.

386 Corollary Let $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then

$$\cos((2k+1)\pi + x) = -\cos x \tag{9.12}$$

and

$$\sin((2k+1)\pi + x) = -\sin x \tag{9.13}$$

In other words, if we add even multiples of π to a real number, we get back the same cosine and the sine of the real number. If we add odd multiples of π to a real number, we get minus the cosine or sine of the real number.

387 Example Write

$$\sin(32\pi + x) - 18\cos(19\pi - x) + \cos(56\pi + x) - 9\sin(x + 17\pi)$$

in the form $a\sin x + b\cos x$.

Solution: ► *The even multiples of π addends give*

$$\sin(32\pi + x) = \sin x$$

and

$$\cos(56\pi + x) = \cos x.$$

Examining the odd multiples of π addends we see that $\cos(19\pi - x) = -\cos(-x)$. *But* $\cos(-x) = \cos x$*, as the cosine is an even function and so*

$$\cos(19\pi - x) = -\cos x.$$

Similarly,

$$\sin(17\pi + x) = -\sin x.$$

Upon gathering all of these equalities, we deduce that

$$\begin{aligned} &\sin(32\pi + x) - 18\cos(19\pi - x) \\ &\quad + \cos(56\pi + x) - 9\sin(x + 17\pi) \\ &= \sin x - 18(-\cos x) \\ &\quad + \cos x - 9\sin x \\ &= -8\sin x + 19\cos x. \end{aligned}$$

◄

388 Example Prove that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Solution: ► *$\mathscr{C}\left(\frac{\pi}{4}\right)$ is half-way between $\mathscr{C}(0)$ and $\mathscr{C}\left(\frac{\pi}{2}\right)$. Thus $\triangle OCM$ in figure 9.9 is an isosceles right triangle. As $OC = CM$, we have*

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4}.$$

By the Pythagorean Relation,

$$\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4} = 1,$$

and so $2\cos^2\frac{\pi}{4} = 1$. Since $\mathscr{C}\left(\frac{\pi}{4}\right)$ lies in the first quadrant, we take the positive square root. We deduce $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. This implies that $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. ◄

389 Example Prove that $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$ and that $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

Solution: ► *In figure 9.10 , $A = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$, $B = (0,0)$ and $C = (1,0)$. Since $BA = BC = 1$, $\triangle BAC$ is isosceles. Thus $\angle A = \angle C$. Moreover, since the sum of the angles of a triangle is π radians and central $\angle B$ measures $\dfrac{\pi}{3}$ radians, the triangle is equilateral. Let D denote the foot of the perpendicular from A to the side BC. Since $\triangle BAC$ is equilateral, D is halfway of the distance between B and C, which means that $= \dfrac{1}{2}$. Thus*

$$\cos\frac{\pi}{3} = \frac{1}{2}.$$

Also, taking the positive square root (why?)

$$\sin\frac{\pi}{3} = \sqrt{1-\cos^2\frac{\pi}{3}} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2},$$

as we wanted to shew. ◄

390 Example Prove that $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ and that $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$.

Solution: ► *Reflect the point $A = (\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$ about the x-axis to the point $C = (\cos\frac{\pi}{6}, -\sin\frac{\pi}{6})$, as in figure 9.11. Observe that since $\angle DBA = \angle CBD = \dfrac{\pi}{6}$ then $\angle CBA = \dfrac{\pi}{3}$. Thus $\triangle ABC$ is equilateral, and so $AD = \frac{1}{2}$, which implies that*

$$\sin\frac{\pi}{6} = \frac{1}{2}.$$

We deduce that

$$\cos\frac{\pi}{6} = \sqrt{1-\sin^2\frac{\pi}{6}} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}.$$

◄

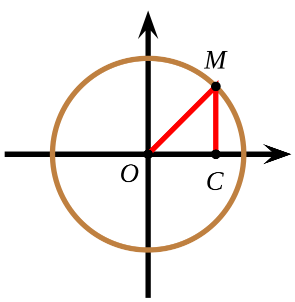

Figure 9.9: $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$

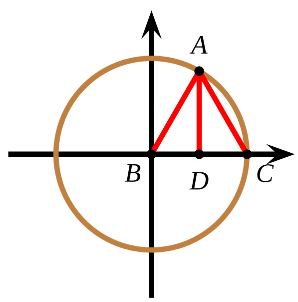

Figure 9.10: $\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}$

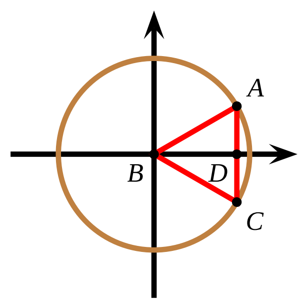

Figure 9.11: $\sin\frac{\pi}{6}$ and $\cos\frac{\pi}{6}$

The student will do well in memorising the special values deduced above, which are conveniently gathered in the table below.

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

391 Example Find $\cos(-\frac{\pi}{6})$ and $\sin(-\frac{\pi}{6})$.

Solution: ► *Since* $x \mapsto \cos x$ *is an even function, we have*

$$\cos(-\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}.$$

Since $x \mapsto \sin x$ *is an odd function, we have*

$$\sin(-\frac{\pi}{6}) = -\sin(\frac{\pi}{6}) = -\frac{1}{2}.$$

◄

392 Example Find $\cos\frac{7\pi}{6}$ and $\sin\frac{7\pi}{6}$.

Solution: ► *By the reflexion about the origin identities*

$$\cos\frac{7\pi}{6} = \cos(\pi + \frac{\pi}{6}) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

and

$$\sin\frac{7\pi}{6} = \sin(\pi + \frac{\pi}{6}) = -\sin\frac{\pi}{6} = -\frac{1}{2}.$$

◄

393 Example

$$\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(-\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

and

$$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = -\sin\left(-\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

394 Example Find the exact value of

$$\cos\left(-\frac{32}{3}\pi\right)$$

Solution: ►

$$\begin{aligned} \cos\left(-\tfrac{32}{3}\pi\right) &= \cos\left(\tfrac{32\pi}{3}\right) \\ &= \cos(10\pi + \tfrac{2\pi}{3}) \\ &= \cos(\tfrac{2\pi}{3}) \\ &= -\tfrac{1}{2} \end{aligned}$$

Aliter:

$$\begin{aligned} \cos\left(-\tfrac{32}{3}\pi\right) &= \cos\left(\tfrac{32\pi}{3}\right) \\ &= \cos(11\pi - \tfrac{\pi}{3}) \\ &= -\cos(-\tfrac{\pi}{3}) \\ &= -\cos\left(\tfrac{\pi}{3}\right) \\ &= -\tfrac{1}{2} \end{aligned}$$

◄

395 Example Find the exact value of

$$\sin\left(-\frac{31}{3}\pi\right)$$

Solution: ►

$$\begin{aligned} \sin\left(-\tfrac{31}{3}\pi\right) &= -\sin\left(\tfrac{31\pi}{3}\right) \\ &= -\sin(10\pi + \tfrac{\pi}{3}) \\ &= -\sin(\tfrac{\pi}{3}) \\ &= -\tfrac{\sqrt{3}}{2} \end{aligned}$$

◄

396 Theorem (Complementary Angle Identities) The following identities hold:

$$\cos(\frac{\pi}{2} - x) = \sin x \ \forall x \in \mathbb{R} \tag{9.14}$$

$$\sin(\frac{\pi}{2} - x) = \cos x \ \forall x \in \mathbb{R} \tag{9.15}$$

Proof: *We will prove the result for $x \in [0; \frac{\pi}{2}[$. The extension of these identities to all real numbers depends on Theorem 385 and we leave it as an exercise. In figure 9.12 assume that arc MA (read counterclockwise) measures x and that $x \in [0; \frac{\pi}{4}[$. Reflect point $A = (\cos x, \sin x)$ about the line $y = x$, to point $B = (\sin x, \cos x)$ as in figure 9.12. Arc BT (read counterclockwise) measures x, and so arc MAB measures $\frac{\pi}{2} - x$. This means that $B = (\cos(\frac{\pi}{2} - x), \sin(\frac{\pi}{2} - x))$, from where the Theorem follows for $x \in]0; \frac{\pi}{4}[$. Assume now that $x \in [\frac{\pi}{4}; \frac{\pi}{2}[$. Then $\frac{\pi}{2} - x \in [0; \frac{\pi}{4}[$, and so we apply the result just obtained to $\frac{\pi}{2} - x$:*

$$\cos\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = \sin x,$$

and

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = \cos x.$$

So, we have established the result for $x \in [0; \frac{\pi}{2}[$. ❑

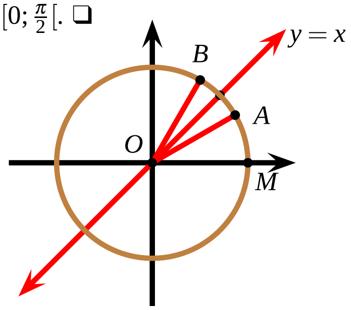

Figure 9.12: Complementary Angle Identities.

Using the complementary angle identities,

$$\sin\frac{\pi}{6} = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2},$$

and

$$\cos\frac{\pi}{6} = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

for instance.

397 Example Prove that

$$\sin x = \cos\left(x - \frac{\pi}{2}\right), \ \forall x \in \mathbb{R}.$$

Solution: ► *Since the cosine is an even function,*

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) = \cos\left(-\left(\frac{\pi}{2} - x\right)\right) = \cos\left(x - \frac{\pi}{2}\right).$$

◄

398 Example Prove that the following hold identically.

$$\cos x = \sin\left(x + \frac{\pi}{2}\right), \ \forall x \in \mathbb{R}.$$

$$-\sin x = \cos\left(x + \frac{\pi}{2}\right), \ \forall x \in \mathbb{R}.$$

Solution: ► *Using the fact the fact that the cosine is an even function, and using the complementary angle identity for the cosine,*

$$\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} - (-x)\right) = \sin\left(\frac{\pi}{2} + x\right).$$

Since the sine is an odd function,

$$\sin x = -\sin(-x) = -\cos\left(\frac{\pi}{2} - (-x)\right) = -\cos\left(\frac{\pi}{2} + x\right).$$

◄

399 Example Let $0 < \theta < \frac{\pi}{2}$. Given that $\sin 2\theta = \cos 3\theta$ find $\sin 5\theta$.

Solution: ► *Since* $\sin 2\theta = \cos 3\theta$*, these two quantities have the same sign. Since* $0 < 2\theta < \pi$*, then both* $\mathscr{C}(2\theta)$ *and* $\mathscr{C}(3\theta)$ *must be in quadrant I. By the complementary angle identities, we have* $\sin 2\theta = \cos(\frac{\pi}{2} - 2\theta)$*. Thus* $\cos(\frac{\pi}{2} - 2\theta) = \cos 3\theta$*, and so,* $\frac{\pi}{2} - 2\theta = 3\theta$ *or* $5\theta = \frac{\pi}{2}$*. Hence* $\sin 5\theta = 1$. ◄

400 Example Write in the form $a \sin\alpha + b \sin\alpha$:

$$\sin(\pi - \alpha) + \cos\left(\frac{\pi}{2} + \alpha\right) - \cos(\pi + \alpha)$$

Solution: ► *By reflexion about the origin,* $\sin(\pi - \alpha) = -\sin(-\alpha)$*. Since the sine is an odd function,* $-\sin(-\alpha) = -(-\sin\alpha) = \sin\alpha$*. By the complementary angle identities, and since the sine is an odd function*

$$\cos\left(\frac{\pi}{2} + \alpha\right) = \cos\left(\frac{\pi}{2} - (-\alpha)\right) = \sin(-\alpha) = -\sin\alpha.$$

Finally, by reflexion about the origin, $\cos(\pi + \alpha) = -\cos\alpha$*. Upon collecting all of these equalities,*

$$\sin(\pi - \alpha) + \cos\left(\frac{\pi}{2} + \alpha\right) - \cos(\pi + \alpha) = \cos\alpha.$$

◄

401 Example Given that

$$3\sin x + 4\cos x = 5,$$

find $\sin x$ and $\cos x$.

Solution: ► *We have*

$$3\sin x + 4\cos x = 5 \iff \sin x = \frac{5-4\cos x}{3}.$$

Putting this in the identity $\cos^2 x + \sin^2 x = 1$ *we obtain*

$$\cos^2 x + \left(\frac{5-4\cos x}{3}\right)^2 = 1$$

$$\cos^2 x + \frac{25-40\cos x + 16\cos^2 x}{9} = 1$$

$$9\cos^2 x + 25 - 40\cos x + 16\cos^2 x = 9$$

$$25\cos^2 x - 40\cos x + 16 = 0$$

$$(5\cos x - 4)^2 = 0$$

$$\cos x = \frac{4}{5}$$

Substituting this value we obtain

$$\sin x = \frac{5-4\cos x}{3} = \frac{5-\frac{16}{5}}{3} = \frac{3}{5}.$$

◄

402 Example Find k such that the expression

$$(\sin x + \cos x)^2 + k\sin x\cos x = 1$$

becomes an identity.

Solution: ► *We have*

$$\begin{aligned} 1 &= (\sin x + \cos x)^2 + k\sin x\cos x \\ &= \sin^2 x + 2\sin x\cos x + \cos^2 x + k\sin x\cos x \\ &= 1 + (k+2)\sin x\cos x \end{aligned}$$

We thus have $(k+2)\sin x\cos x = 0$. *This will hold for all real numbers* x *if* $k = -2$. ◄

Homework

9.2.1 Problem Write in the form $a\sin x + b\cos x$, with real constants a, b.

$$A(x) = \sin\left(\frac{\pi}{2} - x\right) + \cos(5\pi - x) + \cos\left(\frac{3\pi}{2} - x\right) + \sin\left(\frac{3\pi}{2} + x\right)$$

9.2.2 Problem True or False.

1. $\sin\frac{7\pi}{6} = 1/2$.
2. $\cos(\frac{\pi}{2} + 99) = \sin 99$.
3. $\cos(-1993) = \cos 1993$.
4. $\sin(-1993) = -\sin 1993$.
5. If $\sin x = 1$, then $x = \pi/2$.
6. $\cos(\cos\pi) = \cos(\cos 0)$.
7. $\forall x \in \mathbb{R},\ \sin 2x = 2\sin x$.
8. $\exists x \in \mathbb{R}$ such that $\cos x = 2$.
9. $\exists x \in \mathbb{R}$ such that $\cos^2 x = \cos x^2$
10. $(\sin x + \cos x)^2 = 1,\ \forall x \in \mathbb{R}$.
11. $\cos x = \sin(x + \frac{\pi}{2}),\ \forall x \in \mathbb{R}$.
12. $\sin x = \cos(x - \frac{\pi}{2}),\ \forall x \in \mathbb{R}$.

13. $\sin x = \cos(x + \frac{\pi}{2}), \ \forall x \in \mathbb{R}$.
14. $-\frac{1}{2} \leq \cos\frac{x}{2} \leq \frac{1}{2}, \ \forall x \in \mathbb{R}$.
15. $1 \leq -2\cos\frac{x}{2} + 3 \leq 5, \ \forall x \in \mathbb{R}$.
16. $\exists A \in \mathbb{R}$ such that the equation $\cos x = A$ has exactly 7 real solutions.
17. $\cos^2 x - \sin^2 x = -1, \forall x \in \mathbb{R}$.

9.2.3 Problem Given that $\sin t = -0.8$ and $\mathscr{C}(t)$ lies in the fourth quadrant, find $\cos t$.

9.2.4 Problem Given that $\cos u = -0.9$ and $\mathscr{C}(u)$ lies in the second quadrant, find $\sin u$.

9.2.5 Problem Given that $\sin t = \frac{\sqrt{7}}{5}$ and $\mathscr{C}(t)$ lies in the first quadrant, find $\cos t$.

9.2.6 Problem Given that $\cos u = \frac{\sqrt{13}}{4}$ and $\mathscr{C}(u)$ lies in the third quadrant, find $\sin u$.

9.2.7 Problem Using the fact that $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, find $\cos\frac{5\pi}{6}$ and $\sin\frac{5\pi}{6}$.

9.2.8 Problem Using the fact that $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$, find $\cos\frac{3\pi}{4}$ and $\sin\frac{3\pi}{4}$.

9.2.9 Problem Find $\sin(\dfrac{31\pi}{6})$ and $\cos(\dfrac{31\pi}{6})$.

9.2.10 Problem Find $\sin(\dfrac{20\pi}{3})$ and $\cos(\dfrac{20\pi}{3})$.

9.2.11 Problem Find $\sin(\dfrac{17\pi}{4})$ and $\cos(\dfrac{17\pi}{4})$.

9.2.12 Problem Find $\sin(\dfrac{-15\pi}{4})$ and $\cos(-\dfrac{15\pi}{4})$.

9.2.13 Problem Find $\sin(\dfrac{202\pi}{3})$ and $\cos(\dfrac{202\pi}{3})$.

9.2.14 Problem Find $\sin(\dfrac{171\pi}{4})$ and $\cos(\dfrac{171\pi}{4})$.

9.2.15 Problem If $|\sin\theta| < 1$ and $|\cos\theta| > 0$, prove that

$$\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} = \frac{2}{\cos\theta}$$

holds identically.

9.2.16 Problem Given that

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4},$$

find $\sin\frac{2\pi}{5}$, $\cos\frac{3\pi}{5}$ and $\sin\frac{3\pi}{5}$

9.2.17 Problem Given that $\cos\alpha + \sin\alpha = A$ and $\sin\alpha\cos\alpha = B$, prove that $A^2 - 2B = 1$

9.2.18 Problem Given that $\cos\alpha + \sin\alpha = A$ and $\sin\alpha\cos\alpha = B$, prove that $\sin^3\alpha + \cos^3\alpha = A - AB$.

9.2.19 Problem Demonstrate that for all real numbers x, the following is an identity

$$(\sin x + 4\cos x)^2 + (4\sin x - \cos x)^2 = 17$$

9.2.20 Problem Prove that $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x$ is an identity.

9.2.21 Problem Prove that

$$\sqrt{1 + 2\sin x\cos x} = |\sin x + \cos x|, \quad \forall x \in \mathbb{R}.$$

9.2.22 Problem Prove that $\forall x \in \mathbb{R}$,

$$\sin^4 x + \cos^4 x + 2(\sin x\cos x)^2 = 1.$$

9.2.23 Problem Prove, by recurrence, that

$$\sin(x + n\pi) = (-1)^n \sin x,$$

and

$$\cos(x + n\pi) = (-1)^n \cos x.$$

9.2.24 Problem Prove that $\forall x \in \mathbb{R}$,

$$\sin^6 x + \cos^6 x + 3(\sin x\cos x)^2 = 1.$$

9.2.25 Problem Prove that

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \frac{\sin x + 1}{\cos x}$$

$\forall x \in \mathbb{R}$ such that $\sin x + \cos x \neq 1$ and $\cos x \neq 0$.

9.2.26 Problem (AHSME 1976) If $\sin x + \cos x = \frac{1}{5}$ and $x \in]0;\pi[$, find $\cos x$ and $\sin x$.

9.2.27 Problem (AIME 1983) Find the minimum value of the function

$$x \mapsto \frac{9x^2\sin^2 x + 4}{x\sin x}$$

over the interval $]0;\pi[$.

9.3 The Graphs of Sine and Cosine

To obtain the graph of $x \mapsto \sin x$, we traverse the circumference of the unit circle, starting from $(1, 0)$, in a levogyrate (counterclockwise) sense, recording each time the abscissa of the point visited. See figure 9.13.

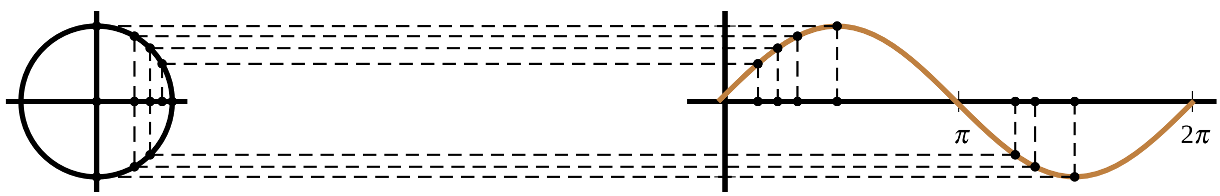

Figure 9.13: The graph $y = \sin x$ for $x \in [0; 2\pi[$.

Since $x \mapsto \sin x$ is periodic with period 2π and an odd function, we may now graph $x \mapsto \sin x$ for all values of x. See figure 9.14.

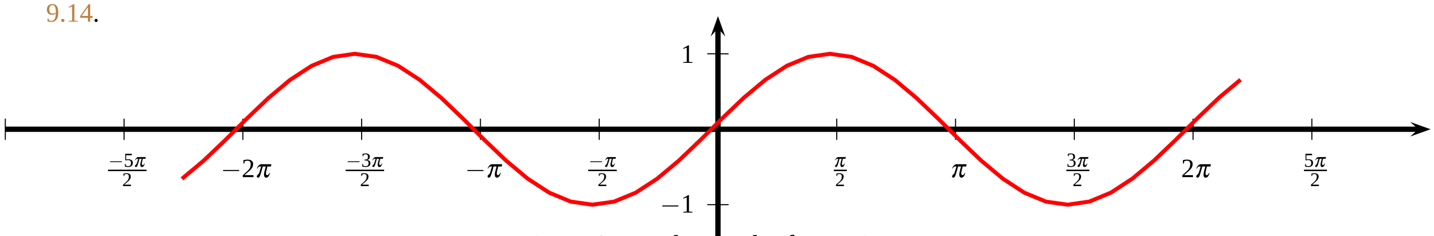

Figure 9.14: The graph of $x \mapsto \sin x$.

403 Example (Jordan's Inequality) Give a graphical argument justifying the inequality $\frac{2}{\pi}x \leq \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.

Solution: ► *The equation of the straight line joining* $(0, 0)$ *and* $(\frac{\pi}{2}, 1)$ *is* $y = \frac{2}{\pi}x$. *From the graphs below, the graph of* $y = \frac{2}{\pi}x$ *lies below that of* $y = \sin x$ *in the interval* $[0; \frac{\pi}{2}]$. *See figure 9.15.* ◄

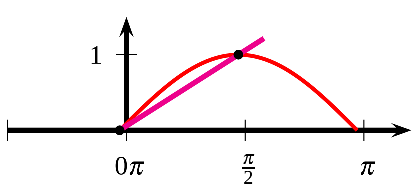

Figure 9.15: Jordan's Inequality.

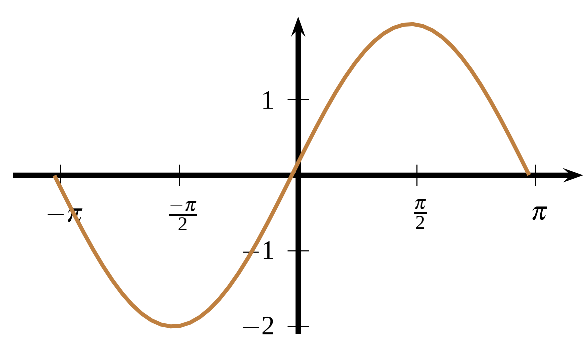

Figure 9.16: The graph of $x \mapsto 2\sin x$.

404 Example Graph $x \mapsto 2\sin x$.

Solution: ► *Recall that if* $y = f(x)$, *then* $y = 2f(x)$ *is a distortion of the graph of* $y = f(x)$, *in which the y-coordinate is doubled. The graph of* $x \mapsto 2\sin x$ *is shewn in figure 9.16. Observe that* $-2 \leq 2\sin x \leq 2$, *so the least value that* $x \mapsto 2\sin x$ *could attain is* -2 *and the largest value is* 2. ◄

405 Definition The average between the least value and largest value of of a periodic function its *amplitude.*

406 Theorem Let $A \in \mathbb{R} \setminus \{0\}$. Then $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \sin Ax \end{array}$ and $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \cos Ax \end{array}$ have period $\dfrac{2\pi}{|A|}$.

Proof: *Since $x \mapsto \sin x$ and $x \mapsto \cos x$ have period 2π, then, if $A \in \mathbb{R} \setminus \{0\}$ is constant, we have*

$$\sin A\left(x + \frac{2\pi}{|A|}\right) = \sin(Ax \pm 2\pi) = \sin Ax,$$

and

$$\cos A\left(x + \frac{2\pi}{|A|}\right) = \cos(Ax \pm 2\pi) = \cos Ax,$$

whence $x \mapsto \sin Ax$ and $x \mapsto \cos Ax$ have period at most $\frac{2\pi}{|A|}$.

Could the period of $x \mapsto \sin Ax, A \neq 0$ and $x \mapsto \cos Ax, A \neq 0$ be smaller than $\dfrac{2\pi}{|A|}$*? Assume $0 < P < \dfrac{2\pi}{|A|}$ is a period for these functions. Then $0 < P|A| < 2\pi$ and $\sin Ax = \sin A(x \pm P)$ and $\cos Ax = \cos A(x \pm P)$. In particular,*

$$0 = \sin 0 = \sin \pm AP.$$

This means that $|A|P$ is a zero of $x \mapsto \sin x$. Since $0 < |A|P < 2\pi$, we must have $|A|P = \pi$. Now

$$1 = \cos 0 = \cos \pm AP = \cos |A|P = \cos \pi = -1,$$

a contradiction. Thus the period of $x \mapsto \sin Ax, A \neq 0$ and $x \mapsto \cos Ax, A \neq 0$ is precisely $\dfrac{2\pi}{|A|}$, as we wanted to shew. ❑

407 Example Graph $x \mapsto \sin 2x$.

Solution: ► *Since $-1 \leq \sin 2x \leq 1$, the amplitude of $x \mapsto \sin 2x$ is $\frac{1-(-1)}{2} = 1$. The period of $x \mapsto \sin 2x$ is $2\pi \div 2 = \pi$. Recall that if $y = f(2x)$, then $y = f(2x)$ is a distortion of the graph of $y = f(x)$, in which the x-coordinate is halved. The graph of $x \mapsto \sin 2x$ is shewn in figure 9.17.*

◄

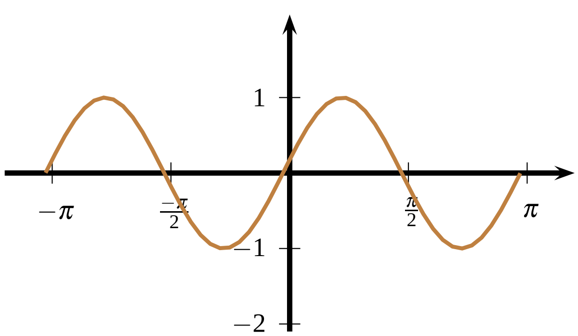

Figure 9.17: The graph of $x \mapsto \sin 2x$.

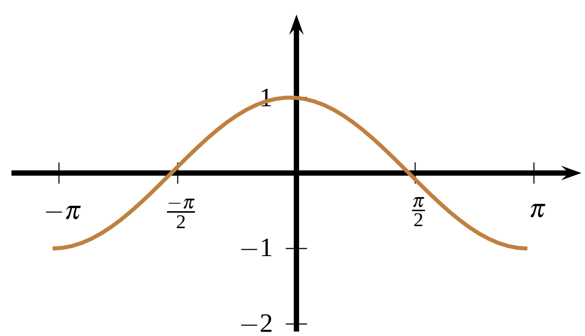

Figure 9.18: The graph of $x \mapsto \cos x$.

408 Example Graph $x \mapsto \sin\left(x + \dfrac{\pi}{2}\right)$

Solution: ► *Recall that if $a > 0$ the graph of $x \mapsto f(x+a)$ is a translation a units to the left of the graph $x \mapsto f(x)$. Now, the cosine is an even function, and by the complementary angle identities, we have*

$$\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} - (-x)\right) = \sin\left(\frac{\pi}{2} + x\right),$$

and so this graph is the same as that of the cosine function. The graph of $y = \sin(x + \frac{\pi}{2}) = \cos x$ is shewn in in figure 9.18. ◄

409 Example Give a purely graphical argument (no calculators allowed!) justifying $\cos 1 < \sin 1$.

Solution: ► *At $x = \frac{\pi}{4}$, the graphs of the sine and the cosine coincide. For $x \in [\frac{\pi}{4}; \frac{\pi}{2}]$, the values of the sine increase from $\frac{\sqrt{2}}{2}$ to 1, whereas the values of the cosine decrease from $\frac{\sqrt{2}}{2}$ to 0. Since $\frac{\pi}{4} < 1 < \frac{\pi}{2}$, we have $\cos 1 < \sin 1$.* ◄

410 Example Graph $x \mapsto -2\cos\dfrac{x}{2} + 3$

Solution: ► *Since $-1 \le \cos\frac{x}{2} \le 1$, we have $1 \le -2\cos\frac{x}{2} + 3 \le 5$. The amplitude of $x \mapsto -2\cos\frac{x}{2} + 3$ is therefore $\frac{5-1}{2} = 2$. The period of $x \mapsto -2\cos\dfrac{x}{2} + 3$ is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$. The graph is shewn in figure 9.19.* ◄

411 Example Draw the graph of $x \mapsto -3\sin\frac{x}{4}$. What is the amplitude, period, and where is the first positive real zero of this function?

Solution: ► *Since $-3 \le -3\sin x \le 3$, the amplitude of $x \mapsto -3\sin\frac{x}{4}$ is $\frac{3-(-3)}{2} = 3$. The period is $2\pi \div \frac{1}{4} = 8\pi$, and the first positive zero occurs when $\frac{x}{4} = \pi$, i.e., at $x = 4\pi$. A portion of the graph is shewn in figure 9.20.* ◄

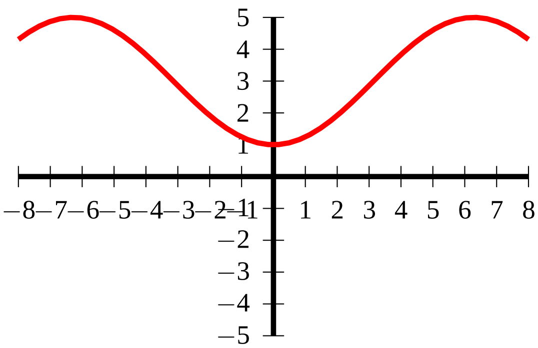

Figure 9.19: The graph of $x \mapsto -2\cos\frac{x}{2} + 3$.

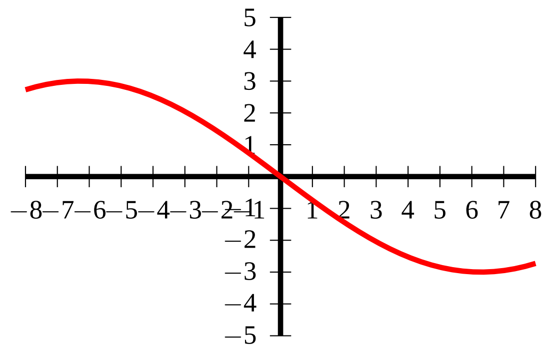

Figure 9.20: The graph of $x \mapsto -3\sin\frac{x}{4}$.

412 Example For which real numbers x is $\log_{\cos x} x$ a real number?

Solution: ► *If $\log_a t$ is defined and real, then $a > 0, a \ne 1$ and $t > 0$. Hence one must have $\cos x > 0, \cos x \ne 1$ and $x > 0$. All this happens when*

$$x \in \left]0; \frac{\pi}{2}\right[\cup \left]\frac{3\pi}{2} + 2\pi n; 2\pi(n+1)\right[\cup \left]2\pi(n+1); \frac{5\pi}{2} + 2\pi n\right[,$$

for $n \ge 0, n \in \mathbb{Z}$. ◄

413 Example For which real numbers x is $\log_x \cos x$ a real number?

Solution: ► *In this case one must have $x > 0, x \ne 1$ and $\cos x > 0$. Hence*

$$x \in \left]0;1\right[\cup \left]1;\frac{\pi}{2}\right[\cup \left]\frac{3\pi}{2}+2\pi n;\frac{5\pi}{2}+2\pi n\right[,$$

for $n \geq 0, n \in \mathbb{Z}$.

◀

414 Example Find the period of $x \mapsto \sin 2x + \cos 3x$.

Solution: ► *Let P be the period of* $x \mapsto \sin 2x + \cos 3x$. *The period of* $x \mapsto \sin 2x$ *is* π *and the period of* $x \mapsto \cos 3x$ *is* $\frac{2\pi}{3}$. *In one full period of length P, both* $x \mapsto \sin 2x$ *and* $x \mapsto \cos 3x$ *must go through an integral number of periods. Thus* $P = s\pi = \frac{2\pi t}{3}$, *for some positive integers s and t. But then* $3s = 2t$. *The smallest positive solutions of this is* $s = 2, t = 3$. *The period sought is then* $P = s\pi = 2\pi$. ◀

415 Example How many real numbers x satisfy

$$\sin x = \frac{x}{100}?$$

Solution: ► *Plainly* $x = 0$ *is a solution. Also, if* $x > 0$ *is a solution, so is* $-x < 0$. *So, we can restrict ourselves to positive solutions.*

If x is a solution then $|x| = 100|\sin x| \leq 100$. *So one can further restrict x to the interval* $]0;100]$. *Decompose* $]0;100]$ *into* 2π*-long intervals (the last interval is shorter):*

$$]0;100] =]0;2\pi] \cup]2\pi;4\pi] \cup]4\pi;6\pi] \cup \cdots \cup]28\pi;30\pi] \cup]30\pi;100].$$

From the graphs of $y = \sin x, y = x/100$ *we see that that the interval* $]0;2\pi]$ *contains only one solution. Each interval of the form* $]2\pi k;\ 2(k+1)\pi], k = 1,2,\ldots,14$ *contains two solutions. As* $31\pi < 100$, *the interval* $]30\pi;100]$ *contains a full wave, hence it contains two solutions. Consequently, there are* $1 + 2\cdot 14 + 2 = 31$ *positive solutions, and hence, 31 negative solutions. Therefore, there is a total of* $31 + 31 + 1 = 63$ *solutions.* ◀

Homework

9.3.1 Problem True or False. Use graphical arguments for the numerical premises. No calculators!

1. $x \mapsto \cos 3x$ has period 3.
2. $\cos 3 > \sin 1$.
3. The first real zero of $x \mapsto 2\sin x + 8$ occurs at $x = \pi$
4. There is a real number x for which the graph of $x \mapsto 8 + \cos \frac{x}{10}$ touches the x-axis.

9.3.2 Problem Graph portions of the following. Find the amplitude, period, and the location of the first positive real zero, if there is one, of each function.

1. $x \mapsto 3\sin x$
2. $x \mapsto \sin 3x$
3. $x \mapsto \sin(-3x)$
4. $x \mapsto 3\sin 3x$
5. $x \mapsto 3\cos x$
6. $x \mapsto \cos 3x$
7. $x \mapsto \frac{1}{3}\cos x$
8. $x \mapsto \cos \frac{1}{3}x$
9. $x \mapsto -2\cos \frac{1}{3}x + 13$
10. $x \mapsto \frac{1}{4}\cos \frac{1}{3}x - 10$
11. $x \mapsto |\sin x|$
12. $x \mapsto \sin|x|$

9.3.3 Problem Find the period of $x \mapsto \sin 3x + \cos 5x$

9.3.4 Problem Find the period of $x \mapsto \sin x + \cos 5x$

9.3.5 Problem How many real solutions are there to

$$\sin x = \log_e x\ ?$$

9.3.6 Problem Let $x \geq 0$. Justify graphically that

$$\sin x \leq x.$$

Your argument must make no appeal to graphing software.

9.3.7 Problem Let $x \in \mathbb{R}$. Justify graphically that

$$1 - \frac{x^2}{2} \leq \cos x.$$

Your argument must make no appeal to graphing software.

9.4 Inversion

Since $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \sin x \end{array}$ is periodic, it is not injective, and hence it does not have an inverse. We can, however, restrict the domain and in this way obtain an inverse of sorts. The choice of the restriction of the domain is arbitrary, but the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ is customarily used.

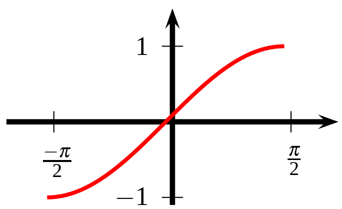

Figure 9.21: $y = \text{Sin } x$

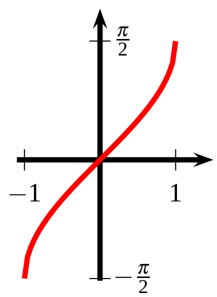

Figure 9.22: $y = \arcsin x$

416 Definition The *Principal Sine Function*, $\begin{array}{ccc} [-\dfrac{\pi}{2};\dfrac{\pi}{2}] & \to & [-1;+1] \\ x & \mapsto & \text{Sin } x \end{array}$ is the restriction of the function $x \mapsto \sin x$ to the interval $[-\dfrac{\pi}{2};\dfrac{\pi}{2}]$. With such restriction

$$\begin{array}{ccc} [-\dfrac{\pi}{2};\dfrac{\pi}{2}] & \to & [-1;+1] \\ x & \mapsto & \text{Sin } x \end{array}$$

is bijective with inverse

$$\begin{array}{ccc} [-1;+1] & \to & [-\dfrac{\pi}{2};\dfrac{\pi}{2}] \\ x & \mapsto & \arcsin x \end{array}$$

The graph of $\begin{array}{ccc} [-1;+1] & \to & [-\dfrac{\pi}{2};\dfrac{\pi}{2}] \\ x & \mapsto & \arcsin x \end{array}$ is thus symmetric with the graph of $\begin{array}{ccc} [-\dfrac{\pi}{2};\dfrac{\pi}{2}] & \to & [-1;+1] \\ x & \mapsto & \text{Sin } x \end{array}$ with respect to the line $y = x$. See figures 9.21 and 9.22 for the graph of $y = \arcsin x$. The notation $\sin^{-1}$ is often used to represent arcsin. The function $x \mapsto \arcsin x$ is an odd function, that is,

$$\arcsin(-x) = -\arcsin x,\ \forall x \in [-1;1].$$

Also, $[-\frac{\pi}{2};\frac{\pi}{2}]$ is the smallest interval containing 0 where all the values of $x \mapsto \text{Sin } x$ in the interval $[-1;1]$ are attained. Moreover, $\forall (x,y) \in [-1;1] \times [-\frac{\pi}{2};\frac{\pi}{2}]$, $y = \arcsin x \iff x = \sin y$.

1. *Whilst it is true that* $\sin\arcsin x = x, \forall x \in [-1;1]$, *the relation* $\arcsin\sin x = x$ *is not always true. For example,* $\arcsin\sin\frac{7\pi}{6} = \arcsin(-\frac{1}{2}) = -\frac{\pi}{6} \neq \frac{7\pi}{6}$.

2. $\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (\arcsin \circ \sin)(x) \end{array}$ *is a* 2π*-periodic odd function with*

$$(\arcsin \circ \sin)(x) = \begin{cases} x & \text{if } x \in \left[0; \frac{\pi}{2}\right] \\ \pi - x & \text{if } x \in \left[\frac{\pi}{2}; \pi\right]. \end{cases}$$

The graph of $x \mapsto (\arcsin \circ \sin)(x)$ *is shewn in figure 9.23.*

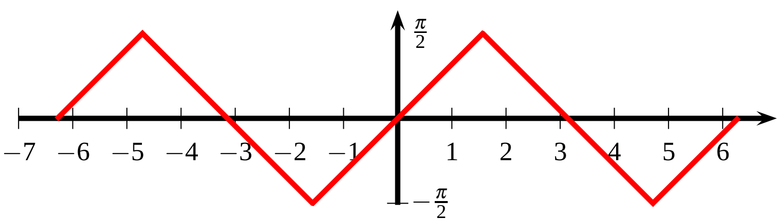

Figure 9.23: $y = (\arcsin \circ \sin)(x)$

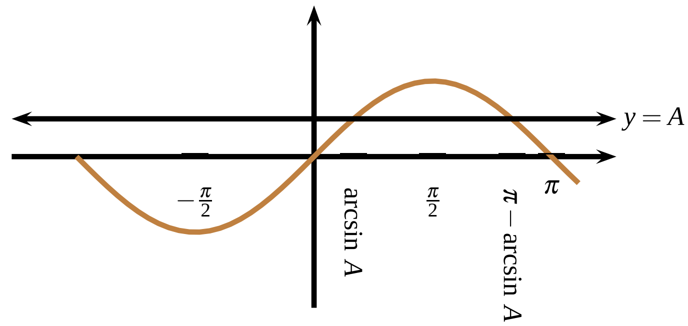

Figure 9.24: The equation $\sin x = A$

417 Theorem The equation

$$\sin x = A$$

has (i) no real solutions if $|A| > 1$, (ii) the infinity of solutions

$$x = (-1)^n \arcsin A + n\pi, \; n \in \mathbb{Z},$$

if $|A| \leq 1$.

Proof: *Since* $-1 \leq \sin x \leq 1$ *for* $x \in \mathbb{R}$*, the first assertion is clear.*

Now, let $|A| \leq 1$*. In figure 9.24 (where we have chosen* $0 \leq A \leq 1$*, the argument for* $-1 \leq A < 0$ *being similar), the first two positive intersections of* $y = A$ *with* $y = \sin x$ *occur at* $x = \arcsin A$ *and* $x = \pi - \arcsin A$*. Since the sine function is periodic with period* 2π*, this means that*

$$x = \arcsin A + 2\pi n, \; n \in \mathbb{Z}$$

and

$$x = \pi - \arcsin A + 2\pi n = -\arcsin A + (2n+1)\pi, \; n \in \mathbb{Z}$$

are the real solutions of this equation. Both relations can be summarised by writing

$$x = (-1)^n \arcsin A + n\pi, \; n \in \mathbb{Z}.$$

This proves the theorem. ❑

418 Example Find all real solutions to $\sin x = -\frac{1}{2}$, and then find all solutions in the interval $[12\pi; \frac{27\pi}{2}]$.

Solution: ► *The general solution to* $\sin x = -\frac{1}{2}$ *is given by*

$$\begin{aligned} x &= (-1)^n \arcsin\left(-\tfrac{1}{2}\right) + n\pi \\ &= (-1)^n\left(-\tfrac{\pi}{6}\right) + n\pi \\ &= (-1)^{n+1}\tfrac{\pi}{6} + n\pi \end{aligned}$$

Now, if

$$12\pi \leq (-1)^{n+1}\frac{\pi}{6} + n\pi \leq \frac{27\pi}{2}$$

then

$$12 - (-1)^{n+1}\frac{1}{6} \leq n \leq \frac{27}{2} - (-1)^{n+1}\frac{1}{6}.$$

The smallest $12-(-1)^{n+1}\frac{1}{6}$ *can be is* $12-\frac{1}{6} = \frac{71}{6} > 11$. *The largest* $\frac{27}{2}-(-1)^{n+1}\frac{1}{6}$ *can be is* $\frac{27}{2}+\frac{1}{6} = \frac{41}{3} < 14$. *So possibly,*

$$11 < n < 14,$$

which means that $n = 12$ *or* $n = 13$.

Testing $n = 12$, $x = -\frac{\pi}{6} + 12\pi = \frac{71\pi}{6}$, *which falls outside the interval and* $x = \frac{\pi}{6} + 13\pi = \frac{79\pi}{6}$, *which falls in the interval. So the only solution in the interval* $[12\pi; \frac{27\pi}{2}]$ *is* $\frac{79\pi}{6}$. ◄

419 Example Find the set of all solutions of

$$\sin\frac{\pi}{x^2} = \frac{1}{2}.$$

Are there any solutions in the interval $]1;3[$?

Solution: ► *We have*

$$\begin{aligned} \frac{\pi}{x^2} &= (-1)^n \arcsin\frac{1}{2} + n\pi = (-1)^n\frac{\pi}{6} + n\pi \\ \frac{1}{x^2} &= (-1)^n\frac{1}{6} + n \\ x^2 &= \frac{1}{(-1)^n\frac{1}{6}+n} \\ x^2 &= \frac{6}{(-1)^n + 6n}. \end{aligned}$$

The expression on the right is negative for integers $n \leq -1$. *Therefore*

$$x = \pm\sqrt{\frac{6}{(-1)^n+6n}}, n = 0,1,2,3,\ldots.$$

The set of all solutions is thus

$$\left\{-\sqrt{\frac{6}{(-1)^n+6n}}, \sqrt{\frac{6}{(-1)^n+6n}}\ n = 0,1,2,3,\ldots\right\}.$$

If

$$1 < \sqrt{\frac{6}{(-1)^n+6n}} < 3,$$

then

$$1 < \frac{6}{(-1)^n+6n} < 9,$$

$$\frac{1}{6} < \frac{1}{(-1)^n + 6n} < \frac{3}{2},$$

$$\frac{2}{3} < 6n + (-1)^n < 6,$$

$$\frac{2}{3} - (-1)^n < 6n < 6 - (-1)^n.$$

The smallest $\frac{2}{3} - (-1)^n$ *can be is* $-\frac{1}{3}$ *and the largest* $6 - (-1)^n$ *can be is* 7. *Hence we must test* n *such that* $-\frac{1}{3} < 6n < 7$, *that is,* $n = 0$ *and* $n = 1$. *If* $n = 0$, *then* $x = \sqrt{6} \in]1;3[$. *If* $n = 1$, *then* $x = \sqrt{\frac{6}{5}} \in]1;3[$. *So the solutions belonging to* $]1;3[$ *are* $x = \sqrt{6}$ *and* $x = \sqrt{\frac{6}{5}}$. ◀

420 Example Find the set of all real solutions to

$$\sin\frac{2}{2x+1} = \frac{\sqrt{2}}{2}$$

Solution: ▶ *We have*

$$\frac{2}{2x+1} = (-1)^n \arcsin\left(\frac{\sqrt{2}}{2}\right) + \pi n, n \in \mathbb{Z},$$

which is equivalent to each of the following equations

$$\frac{2}{2x+1} = (-1)^n \frac{\pi}{4} + \pi n,$$

$$\frac{2x+1}{2} = \frac{1}{(-1)^n \frac{\pi}{4} + \pi n},$$

$$x + \frac{1}{2} = \frac{1}{(-1)^n \frac{\pi}{4} + \pi n},$$

whence the solution set is

$$\left\{-\frac{1}{2} + \frac{4}{(-1)^n \pi + 4n\pi},\ n \in \mathbb{Z}\right\}.$$

◀

421 Example Find the set of all real solutions to

$$2\sin^2 x - \sin x - 1 = 0.$$

Solution: ▶ *Factoring,*

$$0 = 2\sin^2 x - \sin x - 1 = (2\sin x + 1)(\sin x - 1)$$

Hence either $\sin x = -\frac{1}{2}$ *and so*

$$x = (-1)^n \arcsin\frac{1}{2} + \pi n = (-1)^n (\frac{-\pi}{6}) + \pi n = (-1)^{n+1}\frac{\pi}{6} + \pi n,$$

or $\sin x = 1$ *and so*

$$x = (-1)^n \arcsin 1 + \pi n = (-1)^n \frac{\pi}{2} + \pi n.$$

The solution set is therefore

$$\left\{(-1)^{n+1}\frac{\pi}{6} + \pi n,\ (-1)^n \frac{\pi}{2} + \pi n, n \in \mathbb{Z}\right\}.$$

◀

422 Definition The *Principal Cosine Function*, $\begin{array}{ccc} [0;\pi] & \to & [-1;1] \\ x & \mapsto & \text{Cos } x \end{array}$ is the restriction of the function $x \mapsto \cos x$ to the interval $[0;\pi]$. With such restriction

$$\begin{array}{ccc} [0;\pi] & \to & [-1;1] \\ x & \mapsto & \text{Cos } x \end{array}$$

is bijective with inverse

$$\begin{array}{ccc} [-1;1] & \to & [0;\pi] \\ x & \mapsto & \arccos x \end{array}.$$

STOP

1. *The notation* $\cos^{-1}$ *is often used to represent* $\arccos$*.*
2. *Whilst it is true that* $\cos\arccos x = x, \forall x \in [-1;1]$*, the relation* $\arccos\cos x = x$ *is not always true. For example,* $\arccos\cos\frac{7\pi}{6} = \arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6} \neq \frac{7\pi}{6}$*.*
3. $x \mapsto \arccos x$ *is neither an even nor an odd even function.*
4. $\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (\arccos\circ\cos)(x) \end{array}$ *is a* 2π*-periodic even function with*

$$(\arccos\circ\cos)(x) = \begin{cases} x & \text{if } x \in [0;\pi] \\ -x & \text{if } x \in [-\pi;0]. \end{cases}$$

5. $\forall (x,y) \in [-1;1] \times [0;\pi], y = \arccos x \iff x = \cos y$*.*
6. *The graphs of* $x \mapsto \text{Cos } x$ *and* $x \mapsto \arccos x$ *are symmetric with respect to the line* $y = x$*.*

The graph of $x \mapsto \arccos x$ is shewn in figure 9.25.

For convenience, we provide the following table.

x	$\arcsin x$	$\arccos x$	x	$\arcsin x$	$\arccos x$
0	0	$\frac{\pi}{2}$			
1	$\frac{\pi}{2}$	0	-1	$-\frac{\pi}{2}$	π
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$

423 Theorem The equation

$$\cos x = A$$

has (i) no real solutions if $|A| > 1$, (ii) the infinity of solutions

$$x = \pm \arccos A + 2n\pi, \; n \in \mathbb{Z},$$

if $|A| \leq 1$.

Proof: *Since* $-1 \leq \cos x \leq 1$ *for* $x \in \mathbb{R}$, *the first assertion is clear. Now, let* $|A| \leq 1$. *In figure 9.26 (where we have chosen* $0 \leq A \leq 1$, *the argument for* $-1 \leq A < 0$ *being similar), the two intersections of* $y = A$ *with* $y = \cos x$ *closest to* $x = 0$ *occur at* $x = \arccos A$ *and* $x = -\arccos A$. *Since the cosine function is periodic with period* 2π, *this means that*

$$x = \arccos A + 2\pi n, \; n \in \mathbb{Z}$$

and

$$x = -\arccos A + 2\pi n, \; n \in \mathbb{Z}$$

are the real solutions of this equation. Both relations can be summarised by writing

$$x = \pm \arccos A + 2n\pi, \; n \in \mathbb{Z}.$$

This proves the theorem. ❑

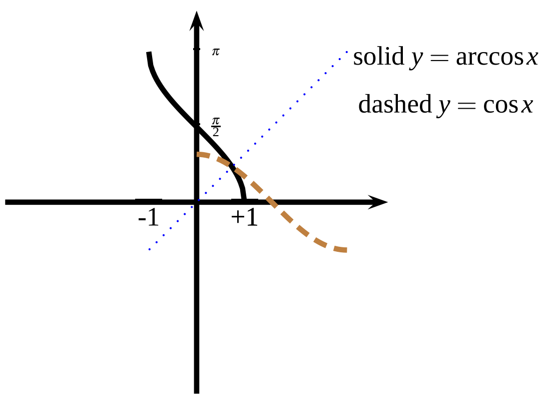

Figure 9.25: $y = \arccos x$

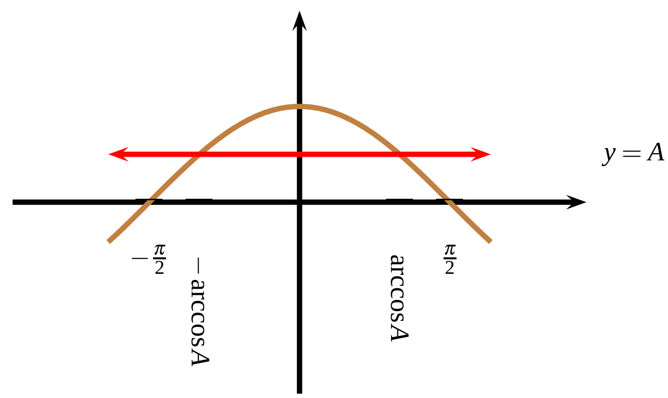

Figure 9.26: The equation $\cos x = A$

424 Example Find the set of all real solutions to

$$2\sin^2 x + 3\cos x - 3 = 0.$$

Solution: ▶ *Since the equation has a cosine to the first power, we write the equation in terms of cosine only, obtaining*

$$\begin{aligned} 0 &= 2\sin^2 x + 3\cos x - 3 \\ &= 2(1-\cos^2 x) + 3\cos x - 3 \\ &= 2\cos^2 x - 3\cos x + 1 \\ &= (2\cos x - 1)(\cos x - 1) \end{aligned}$$

Thus either $\cos x = \frac{1}{2}$, *in which case*

$$x = \pm \arccos \frac{1}{2} + 2\pi n = \pm \frac{\pi}{3} + 2\pi n$$

or $\cos x = 1$ *in which case*

$$x = \pm \arccos 1 + 2\pi n = 2\pi n.$$

The solution set is

$$\left\{ \pm \frac{\pi}{3} + 2\pi n, 2\pi n,\ n \in \mathbb{Z} \right\}.$$

◀

425 Example Find the solutions of the equation

$$\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$$

in the interval $[0; 2\pi]$.

Solution: ▶ *If the logarithmic expression is to make sense, then* $\sqrt{2}\sin x > 0$, $\sqrt{2}\sin x \neq 1$ *and* $1 + \cos x > 0$. *For this we must have*

$$x \in \left]0; \frac{\pi}{4}\right[\cup \left]\frac{\pi}{4}; \frac{3\pi}{4}\right[\cup \left]\frac{3\pi}{4}; \pi\right[.$$

Now, if x belongs to this set

$$\log_{\sqrt{2}\sin x}(1 + \cos x) = 2 \Longleftrightarrow 2\sin^2 x = 1 + \cos x.$$

Using $\sin^2 x = 1 - \cos^2 x$*, the last equality occurs if and only if*

$$(2\cos x - 1)(\cos x + 1) = 0.$$

If $\cos x + 1 = 0$*, then* $x = \pi$*, a value that must be discarded (why?). If* $\cos x = \frac{1}{2}$*, then* $x = \frac{\pi}{3}$*, which is the only solution in* $[0; 2\pi]$ ◀

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426 Example Find the set of all the real solutions to

$$2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7$$

Solution: ▶ *Observe that*

$$\begin{aligned} 2^{\sin^2 x} + 5(2^{\cos^2 x}) - 7 &= 2^{\sin^2 x} + 5(2^{1-\sin^2 x}) - 7 \\ &= 2^{\sin^2 x} + 5(2^1 \cdot 2^{-\sin^2 x}) - 7 \\ &= 2^{\sin^2 x} + \left(\frac{10}{2^{\sin^2 x}}\right) - 7 \\ &= u + \frac{10}{u} - 7. \end{aligned}$$

with $u = 2^{\sin^2 x}$*. From this,* $0 = u^2 - 7u + 10 = (u-5)(u-2)$*. Thus either* $u = 2$*,, meaning* $2^{\sin^2 x} = 2$ *which is to say* $\sin x = \pm 1$ *or* $x = (-1)^n(\frac{\pm\pi}{2}) + n\pi$*. When* $2^{\sin^2 x} = 5$ *one sees that* $\sin^2 x = \log_2 5$*. Since the sinistral side of the last equality is at most 1 and its dextral side is greater than 1, there are no real roots in this instance. The solution set is thus*

$$\left\{ (-1)^n(\frac{\pm\pi}{2}) + n\pi,\ n \in \mathbb{Z} \right\}.$$

◀

427 Example Find all the real solutions of the equation

$$\cos^{2000} x - \sin^{2000} x = 1.$$

Solution: ► *Transposing*

$$\cos^{2000} x = \sin^{2000} x + 1.$$

The dextral side is ≥ 1 *and the sinistral side is* ≤ 1. *Thus equality is only possible if both sides are equal to* 1, *which entails that* $\cos x = 1$ *or* $\cos x = -1$, *whence* $x = \pi n, n \in \mathbb{Z}$. ◄

428 Example Find all the real solutions of the equation

$$\cos^{2001} x - \sin^{2001} x = 1.$$

Solution: ► *Since* $|\cos x| \leq 1$ *and* $|\sin x| \leq 1$, *we have*

$$\begin{aligned} 1 &= \cos^{2001} x - \sin^{2001} x \\ &= \cos^{2001}(-x) + \sin^{2001}(-x) \\ &\leq |\cos^{2001}(-x)| + |\sin^{2001}(-x)| \\ &= |\cos^{1999}(-x)|\cos^2(-x) + |\sin^{1999}(-x)|\sin^2(-x) \\ &\leq \cos^2(-x) + \sin^2(-x) \\ &= 1. \end{aligned}$$

The inequalities are tight, and so equality holds throughout. The first inequality above is true if and only if $\cos(-x) \geq 0$ *and* $\sin(-x) \geq 0$. *The second inequality is true if and only if* $|\cos(-x)| = 1$ *or* $|\sin(-x)| = 1$. *Hence we must have either* $\cos(-x) = 1$ *or* $\sin(-x) = 1$.*This means* $x = 2n\pi$ *or* $x = -\frac{\pi}{2} + 2n\pi$ *where* $n \in \mathbb{Z}$. ◄

429 Example What is $\sin \arccos \frac{3}{4}$?

Solution: ► *Put* $t = \arccos \frac{3}{4}$. *Then* $\frac{3}{4} = \cos t$ *with* $t \in [0; \frac{\pi}{2}]$. *In the interval* $[0; \frac{\pi}{2}]$, $\sin t$ *is positive. Hence*

$$\sin t = \sqrt{1 - \cos^2 y} = \sqrt{1 - \left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}.$$

◄

430 Example What is $\sin \arccos(-\frac{3}{7})$?

Solution: ► *Put* $t = \arccos(-\frac{3}{7})$. *Then* $-\frac{3}{7} = \cos t$ *with* $y \in [\frac{\pi}{2}; \pi]$. *In the interval* $[\frac{\pi}{2}; \pi]$, $\sin t$ *is positive. Hence*

$$\sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - \left(-\frac{3}{7}\right)^2} = \frac{2\sqrt{10}}{7}.$$

◄

431 Example Let $x \in]-\frac{1}{5}; 0[$. Express $\sin \arccos 5x$ as a function of x.

Solution: ► *First notice that* $5x \in]-1; 0[$, *which means that* $\arccos 5x \in]\frac{\pi}{2}; \pi[$, *an interval where the sine is positive. Put* $t = \arccos 5x$. *Then* $5x = \cos t$. *Finally,*

$$\sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - 25x^2}.$$

◄

432 Example Prove that

$$\arcsin x + \arccos x = \frac{\pi}{2}, \forall x \in [-1;1].$$

Solution: ► *By the complementary angle identity for the cosine,*

$$\cos\left(\frac{\pi}{2} - \arcsin x\right) = \sin(\arcsin x) = x.$$

Since $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$, *we have* $\frac{\pi}{2} - \arcsin x \in [0;\pi]$. *This means that*

$$\cos\left(\frac{\pi}{2} - \arcsin x\right) = x \iff \frac{\pi}{2} - \arcsin x = \arccos x,$$

whence the desired result follows. ◄

Homework

9.4.1 Problem True or False.

1. $\arcsin\frac{\pi}{2} = 1$.
2. If $\arccos x = -\frac{1}{2}$, then $x = -\frac{\pi}{3}$.
3. If $\arcsin x \geq 0$ then $x \in [0;\frac{\pi}{2}]$.
4. $\arccos\ \cos(-\frac{\pi}{3}) = \frac{\pi}{3}$.
5. $\arccos\ \cos(-\frac{\pi}{6}) = -\frac{\pi}{6}$.
6. $\arcsin\frac{1}{2000} + \arccos\frac{1}{2000} = \frac{\pi}{2}$.
7. $\exists x \in \mathbb{R}$ such that $\arcsin x > 1$.
8. $-1 \leq \arccos x \leq 1,\ \forall x \in \mathbb{R}$.
9. $\sin\arcsin x = x,\ \forall x \in \mathbb{R}$.
10. $\arccos(\cos x) = x,\ \forall x \in [0;\pi]$.

9.4.2 Problem Find all the real solutions to $2\sin x + 1 = 0$ in the interval $[-\pi;\pi]$.

9.4.3 Problem Find the set of all real solutions to

$$\sin\left(3x - \frac{\pi}{4}\right) = 0.$$

9.4.4 Problem Find the set of all real solutions of the equation

$$-2\sin^2 x - \cos x + 1 = 0.$$

9.4.5 Problem Find all the real solutions to $\sin 3x = -1$. Find all the solutions belonging to the interval $[98\pi;100\pi]$.

9.4.6 Problem Find the set of all real solutions to

$$5\cos^2 x - 2\cos x - 7 = 0.$$

9.4.7 Problem Find the set of all real solutions to

$$\sin x \cos x = 0.$$

9.4.8 Problem Find the set of all real solutions to

$$\cos 3x = \frac{4}{3}.$$

9.4.9 Problem Find the set of all real solutions to

$$4\sin^2 2x - 3 = 0.$$

9.4.10 Problem Find all real solutions belonging to the interval $[-2;2]$, if any, to the following equations.

1. $4\sin^2 x - 3 = 0$
2. $2\sin^2 x = 1$
3. $\cos\frac{2x}{3} = -\frac{\sqrt{3}}{2}$
4. $\sin\frac{3}{x} = 1$
5. $\dfrac{1+\sin x}{1-\cos x} = 0$

9.4.11 Problem Find $\sin\arccos\frac{1}{3}$.

9.4.12 Problem Find $\cos\arcsin(-\frac{2}{3})$.

9.4.13 Problem Find $\sin\arccos(-\frac{2}{3})$.

9.4.14 Problem Find $\arcsin(\sin 5)$; $\arccos(\cos 10)$

9.4.15 Problem Find all the real solutions of the following equations.

1. $\cos x + \dfrac{1}{\cos x} = \frac{3}{2}$.
2. $2\cos^3 x + \cos^2 x - 2\cos x - 1 = 0$.
3. $6\cos^2\left(5x - \dfrac{\pi}{3}\right) - \cos\left(5x - \dfrac{\pi}{3}\right) = 2$.
4. $4\cos^2 x - 2(\sqrt{2}+1)\cos x + \sqrt{2} = 0$.
5. $4\cos^4 x - 17\cos^2 x + 4 = 0$.
6. $(2\cos x + 1)^2 - 4\cos^2 x + (\sin x)(2\cos x + 1) + 1 = 0$.
7. $4\sin^2 x - 2(\sqrt{3} - \sqrt{2})\sin x = \sqrt{6}$.
8. $-2\sin^2 x + 19|\sin x| + 10 = 0$.

9.4.16 Problem Demonstrate that

$$\arccos\ x + \arccos(-x) = \pi,\ \ \forall x \in [-1;1],$$

$$\arcsin x = -\arcsin(-x),\ \ \forall x \in [-1;1].$$

9.4.17 Problem Shew that
$$\arcsin x = \arccos\sqrt{1-x^2},\ \forall x\in[0;1],$$
$$\arccos x = \arcsin\sqrt{1-x^2},\ \forall x\in[0;1].$$

9.4.18 Problem Let $0 < x < \frac{1}{3}$. Find $\cos\left(\arcsin 3x\right)$ and $\cos\left(\arccos 3x\right)$ as functions of x.

9.4.19 Problem Let $-\frac{1}{2} < x < 0$. Find $\sin\left(\arcsin 2x\right)$ and $\sin\left(\arccos 2x\right)$ as functions of x.

9.4.20 Problem Find real constants a, b such that
$$(\arcsin\circ\sin)(x) = ax+b,\ \forall x\in[\frac{99\pi}{2};\frac{101\pi}{2}].$$

9.4.21 Problem Prove that $\begin{array}{ccc}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & (\arccos\circ\cos)(x)\end{array}$ is a 2π-periodic even function and graph a portion of this function for $x\in[-2\pi;2\pi]$.

9.5 The Goniometric Functions

We define the *tangent, secant, cosecant* and *cotangent* of $x\in\mathbb{R}$ as follows.

$$\tan x = \frac{\sin x}{\cos x},\ x\neq\frac{\pi}{2}+\pi n,\ n\in\mathbb{Z}, \tag{9.16}$$

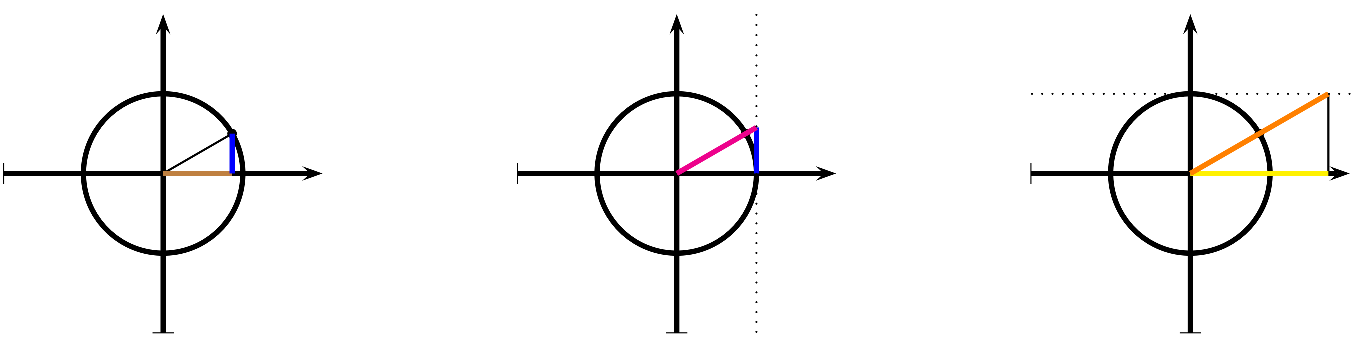
$$\sec x = \frac{1}{\cos x},\ x\neq\frac{\pi}{2}+\pi n,\ n\in\mathbb{Z}, \tag{9.17}$$

$$\csc x = \frac{1}{\sin x},\ x\neq\pi n,\ n\in\mathbb{Z}, \tag{9.18}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x},\ x\neq\pi n,\ n\in\mathbb{Z}. \tag{9.19}$$

The circles below have all radius 1.

cosine		sine	
secant		tangent	
cosecant		cotangent	



1. *The image of* $x\mapsto\tan x$ *over its domain* $\mathbb{R}-\{\frac{\pi}{2}+\pi n,\ n\in\mathbb{Z}\}$ *is* $\mathbb{R}$.
2. *The image of* $x\mapsto\cot x$ *over its domain* $\mathbb{R}-\{\pi n,\ n\in\mathbb{Z}\}$ *is* $\mathbb{R}$.
3. *The image of* $x\mapsto\sec x$ *over its domain* $\mathbb{R}-\{\frac{\pi}{2}+\pi n,\ n\in\mathbb{Z}\}$ *is* $]-\infty;-1]\cup[1;+\infty[$.
4. *The image of* $x\mapsto\csc x$ *over its domain* $\mathbb{R}-\{\pi n,\ n\in\mathbb{Z}\}$ *is* $]-\infty;-1]\cup[1;+\infty[$.

433 Example Given that $\tan x = -3$ and $\mathscr{C}(x)$ lies in the fourth quadrant, find $\sin x$ and $\cos x$.

Solution: ► *In the fourth quadrant* $\sin x < 0$ *and* $\cos x > 0$. *Now,* $-3 = \tan x = \frac{\sin x}{\cos x}$ *entails* $\sin x = -3\cos x$. *As* $\sin^2 x + \cos^2 x = 1$, *One gathers* $9\cos^2 x + \cos^2 x = 1$ *or* $\cos^2 x = \frac{1}{10}$. *Choosing the positive root,* $\cos x = \frac{\sqrt{10}}{10}$. *Finally,*

$$\sin x = -3\cos x = -\frac{3\sqrt{10}}{10}.$$

◄

434 Example Given that $\cot x = 4$ and $\mathscr{C}(x)$ lies in the third quadrant, find the values of $\tan x, \sin x, \cos x, \csc x, \sec x$.

Solution: ► *From* $\cot x = 4$, *we have* $\cos x = 4\sin x$. *Using this and* $\sin^2 x + \cos^2 x = 1$, *we gather* $\sin^2 x + 16\sin^2 x = 1$, *and since* $\mathscr{C}(x)$ *lies in the third quadrant,* $\sin x = -\frac{\sqrt{17}}{17}$. *Moreover,* $\cos x = 4\sin x = -\frac{4\sqrt{17}}{17}$. *Finally,* $\tan x = \frac{1}{\cot x} = \frac{1}{4}$, $\csc x = \frac{1}{\sin x} = -\sqrt{17}$ *and* $\sec x = \frac{1}{\cos x} = -\frac{\sqrt{17}}{4}$. ◄

435 Theorem The function $\begin{array}{ccc} \mathbb{R} - \{\frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\} & \to & \mathbb{R} \\ x & \mapsto & \tan x \end{array}$ is an odd function.

Proof: *If* $x \neq \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}$

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = -\frac{\sin x}{\cos x} = -\tan x,$$

which proves the assertion. ❑

436 Theorem The function $\begin{array}{ccc} \mathbb{R} - \{\frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\} & \to & \mathbb{R} \\ x & \mapsto & \tan x \end{array}$ is periodic with period π.

Proof: *Since*

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \tan x,$$

the period is at most π.

Assume now that $0 < P < \pi$ *is a period for* $x \mapsto \tan x$. *Then* $\tan x = \tan(x+P)\ \forall x \in \mathbb{R}$ *and in particular,*

$$0 = \tan 0 = \tan P = \frac{\sin P}{\cos P},$$

which entails that $\sin P = 0$. *But then* P *would be a positive zero of* $x \mapsto \sin x$ *smaller than* π, *a contradiction. Hence the period of* $x \mapsto \tan x$ *is exactly* π, *which completes the proof.* ❑

How to graph $x \mapsto \tan x$? We start with $x \in [0; \frac{\pi}{2}[$ and then appeal to theorem 435 and theorem 436 to extend this construction for all $x \in \mathbb{R}$.

In figure 9.27, choose B such that the measure of arc AB (measured counterclockwise) be x. Point $A = (1,0)$, and point $B = (\sin x, \cos x)$. Since points B and $(1,t)$ are collinear, the gradient (slope) of the line joining $(0,0)$ and B is the same as that joining $(0,0)$ and $(1,t)$. Computing gradients, we have

$$\frac{\sin x - 0}{\cos x - 0} = \frac{t-0}{1-0},$$

whence $t = \tan x$. We have thus produced a line segment measuring $\tan x$. If we let x vary from 0 to $\pi/2$ we obtain the graph of $x \mapsto \tan x$ for $x \in [0; \frac{\pi}{2}[$.

Since $\cos x = 0$ at $x = \frac{\pi}{2}(2n+1),\ n \in \mathbb{Z}$, $x \mapsto \tan x$ has poles at the points $x = \frac{\pi}{2}(2n+1),\ n \in \mathbb{Z}$. The graph of $x \mapsto \tan x$ is shewn in figure 9.28.

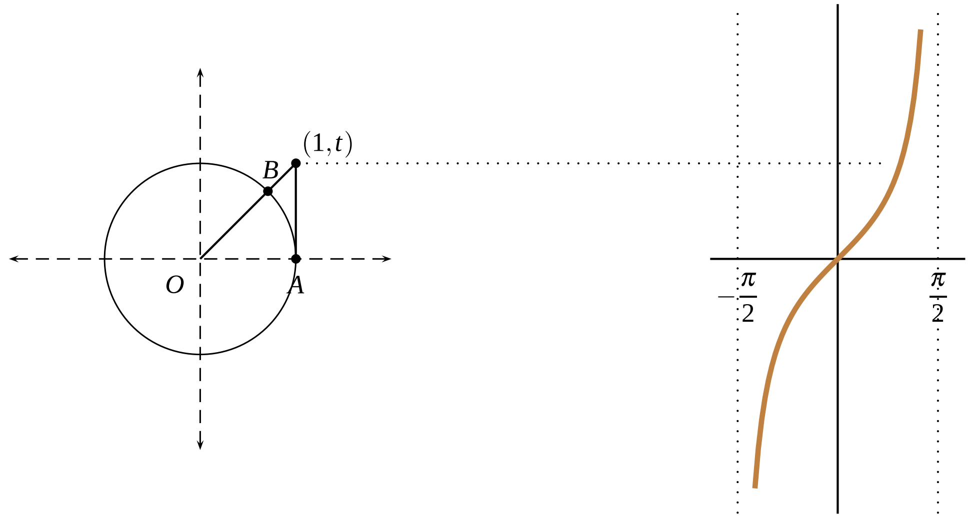


Figure 9.27: Construction of the graph of $x \mapsto \tan x$ for $x \in [0; \frac{\pi}{2}[$.

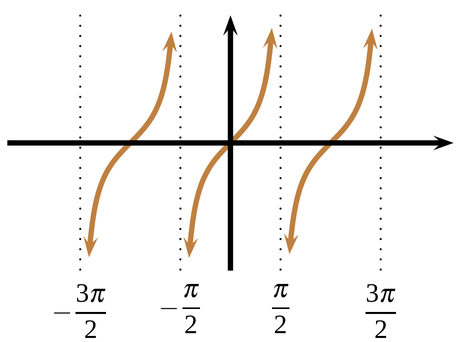


Figure 9.28: $y = \tan x$

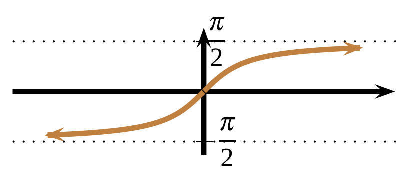


Figure 9.29: $y = \arctan x$

We now define the Principal Tangent function and the arctan function.

437 Definition The *Principal Tangent Function*, $x \mapsto \mathrm{Tan}\ x$ is the restriction of the function $x \mapsto \tan x$ to the interval $]-\dfrac{\pi}{2}; \dfrac{\pi}{2}[$. With such restriction

$$\begin{array}{ccc}]-\dfrac{\pi}{2}; \dfrac{\pi}{2}[& \to & \mathbb{R} \\ x & \mapsto & \mathrm{Tan}\ x \end{array}$$

is bijective with inverse

$$\begin{array}{ccc} \mathbb{R} & \to &]-\dfrac{\pi}{2}; \dfrac{\pi}{2}[\\ x & \mapsto & \arctan x \end{array}$$

The graph of $x \mapsto \arctan x$ is shewn in figure 9.29. Observe that the lines $y = \pm\frac{\pi}{2}$ are asymptotes to $x \mapsto \arctan x$.

1. $\forall x \in \mathbb{R},\ \tan(\arctan(x)) = x$.
2. $\begin{array}{rcl} \mathbb{R} - \{\dfrac{\pi}{2} + n\pi, n \in \mathbb{Z}\} & \to & \mathbb{R} \\ x & \mapsto & (\arctan \circ \tan)(x) \end{array}$ *is an odd π-periodic function.*

438 Theorem The equation

$$\tan x = A,\ A \in \mathbb{R}$$

has the infinitely many solutions

$$x = \arctan A + n\pi, n \in \mathbb{Z}.$$

Proof: *Since the graph of $x \mapsto \tan x$ is increasing in $]-\frac{\pi}{2}; \frac{\pi}{2}[$, it intersects the graph of $y = A$ at exactly one point,*

$$\tan x = A \Longrightarrow x = \arctan A$$

if $x \in]-\frac{\pi}{2}; \frac{\pi}{2}[$. Since $x \mapsto \tan x$ is periodic with period π, each of the points

$$x = \arctan A + n\pi, n \in \mathbb{Z}$$

is also a solution. ❑

439 Example Solve the equation

$$\tan^2 x = 3$$

Solution: ► *Either $\tan x = \sqrt{3}$ or $\tan x = -\sqrt{3}$. This means that $x = \arctan\sqrt{3} + \pi n = \frac{\pi}{3} + \pi n$ or $x = \arctan(-\sqrt{3}) + \pi n = -\frac{\pi}{3} + \pi n$. We may condense this by writing $x = \pm\frac{\pi}{3} + \pi n, n \in \mathbb{Z}$.* ◄

440 Example Solve the equation $(\tan x)^{\sin x} = (\cot x)^{\cos x}$.

Solution: ► *For the tangent and cotangent to be defined, we must have $x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$. Then*

$$(\tan x)^{\sin x} = (\cot x)^{\cos x} = \frac{1}{(\tan x)^{\cos x}}$$

implies

$$(\tan x)^{\sin x + \cos x} = 1.$$

Thus either $\tan x = 1$, in which case $x = \frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$ or $\sin x + \cos x = 0$, which implies $\tan x = -1$, but this does not give real values for the expressions in the original equation. The solution is thus

$$x = \frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}.$$

◄

441 Example Find $\sin\arctan\frac{2}{3}$.

Solution: ► *Put $t = \arctan\frac{2}{3}$. Then $\frac{2}{3} = \tan t, t \in]0; \frac{\pi}{2}[$ and thus $\sin t > 0$. We have $\frac{3}{2}\sin t = \cos t$. As*

$$1 = \cos^2 t + \sin^2 t = \frac{9}{4}\sin^2 t + \sin^2 t,$$

we gather that $\sin^2 t = \frac{4}{13}$. Taking the positive square root $\sin t = \frac{2}{13}$. ◄

442 Example Find the exact value of $\tan\arccos(-\frac{1}{5})$.

Solution: ▶ *Put* $t = \arccos(-\frac{1}{5})$. *As the arccosine of a negative number,* $t \in [\frac{\pi}{2}, \pi]$. *Now,* $\cos t = -\frac{1}{5}$, *and so*

$$\sin t = \sqrt{1 - \left(-\frac{1}{5}\right)^2} = \sqrt{\frac{24}{25}} = \frac{2\sqrt{6}}{5}.$$

We deduce that $\tan t = \frac{\sin t}{\cos t} = -2\sqrt{6}$. ◀

443 Example Let $x \in [0;1[$. Prove that

$$\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}}.$$

Solution: ▶ *Since* $x \in [0;1[$, $\arcsin x \in [0; \frac{\pi}{2}[$. *Put* $t = \arcsin x$, *then* $\sin t = x$, *and* $\cos t > 0$ *since* $t \in [0; \frac{\pi}{2}[$. *Now,* $\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$, *and*

$$\tan t = \frac{\sin t}{\cos t} = \frac{x}{\sqrt{1-x^2}}.$$

Since $t \in [0; \frac{\pi}{2}[$ *this implies that*

$$t = \arctan \frac{x}{\sqrt{1-x^2}},$$

from where the desired equality follows. ◀

444 Theorem The following Pythagorean-like Relation holds.

$$\tan^2 x + 1 = \sec^2 x, \ \forall x \in \mathbb{R} \setminus \{(2n+1)\frac{\pi}{2}, \ n \in \mathbb{Z}\}. \tag{9.20}$$

Proof: *This immediately follows from* $\sin^2 x + \cos^2 x = 1$ *upon dividing through by* $\cos^2 x$. ❑

445 Example Given that $\tan x + \cot x = a$, write $\tan^3 x + \cot^3 x$ as a polynomial in a.

Solution: ▶ *Using the fact that* $\tan x \cot x = 1$, *and the Binomial Theorem:*

$$\begin{aligned}
(\tan x + \cot x)^3 &= \tan^3 x + 3\tan^2 x \cot x + 3\tan x \cot^2 x + \cot^3 x \\
&= \tan^3 x + \sin^3 x + 3\tan x \cot x(\tan x + \cot x) \\
&= \tan^3 x + \sin^3 x + 3(\tan x + \cot x)
\end{aligned}$$

It follows that

$$\tan^3 x + \cot^3 x = (\tan x + \cot x)^3 - 3(\tan x + \cot x) = a^3 - 3a.$$

Aliter: *Observe that* $a^2 = (\tan x + \cot x)^2 = \tan^2 x + \cot^2 x + 2$, *hence* $\tan^2 x + \cot^2 x = a^2 - 2$. *Factorising the sum of cubes*

$$\tan^3 x + \cot^3 x = (\tan x + \cot x)(\tan^2 x - 1 + \cot^2 x) = a(a^2 - 1 - 2)$$

which equals $a^3 - 3a$, *as before.* ◀

446 Example Prove that

$$\frac{2\sin y + 3}{2\tan y + 3\sec y} = \cos y,$$

whenever the expression on the sinistral side be defined.

Solution: ► *Decomposing the tangent and the secant as cosines we obtain,*

$$\begin{aligned}\frac{2\sin y+3}{2\tan y+3\sec y} &= \frac{2\sin y+3}{2\frac{\sin y}{\cos y}+\frac{3}{\cos y}}\\ &= \frac{2\sin y\cos y+3\cos y}{2\sin y+3}\\ &= \frac{(\cos y)(2\sin y+3)}{2\sin y+3}\\ &= \cos y,\end{aligned}$$

as we wished to shew. ◄

447 Example Prove the identity

$$\frac{\tan A+\tan B}{\sec A-\sec B}=\frac{\sec A+\sec B}{\tan A-\tan B},$$

whenever the expressions involved be defined.

Solution: ► *We have*

$$\begin{aligned}\frac{\tan A+\tan B}{\sec A-\sec B} &= \left(\frac{\tan A+\tan B}{\sec A-\sec B}\right)\left(\frac{\tan A-\tan B}{\sec A+\sec B}\right)\left(\frac{\sec A+\sec B}{\tan A-\tan B}\right)\\ &= \left(\frac{\tan^2 A-\tan^2 B}{\sec^2 A-\sec^2 B}\right)\left(\frac{\sec A+\sec B}{\tan A-\tan B}\right)\\ &= \left(\frac{(\sec^2 A-1)-(\sec^2 B-1)}{\sec^2 A-\sec^2 B}\right)\left(\frac{\sec A+\sec B}{\tan A-\tan B}\right)\\ &= \frac{\sec A+\sec B}{\tan A-\tan B},\end{aligned}$$

as we wished to shew. ◄

448 Example Given that $\sin A+\csc A=T$, express $\sin^4 A+\csc^4 A$ as a polynomial in T.

Solution: ► *First observe that*

$$T^2=(\sin A+\csc A)^2=\sin^2 A+\csc^2 A+2\sin A\csc A,$$

hence

$$\sin^2 A+\csc^2 A=T^2-2.$$

By the Binomial Theorem

$$\begin{aligned}T^4 &= (\sin A+\csc A)^4\\ &= \sin^4 A+4\sin^3 A\csc A+6\sin^2 A\csc^2 A+4\sin A\csc^3 A+\csc^4 A\\ &= \sin^4 A+\csc^4 A+6+4(\sin^2 A+\csc^2 A)\\ &= \sin^4 A+\csc^4 A+6+4(T^2-2),\end{aligned}$$

whence $\sin^4 A+\csc^4 A=T^4-4T+2$. ◄

Homework

9.5.1 Problem True or False.

1. $\tan x = \cot \frac{1}{x}$, $\forall x \in \mathbb{R} \setminus \{0\}$.
2. $\exists x \in \mathbb{R}$ such that $\sec x = \frac{1}{2}$.
3. $\arctan 1 = \frac{\arcsin 1}{\arccos 1}$.
4. $x \mapsto \tan 2x$ has period π.

9.5.2 Problem Given that $\csc x = -1.5$ and $\mathscr{C}(x)$ lies on the fourth quadrant, find $\sin x, \cos x$ and $\tan x$.

9.5.3 Problem Given that $\tan x = 2$ and $\mathscr{C}(x)$ lies on the third quadrant, find $\sin x$ and $\cos x$.

9.5.4 Problem Given that $\sin x = t^2$ and $\mathscr{C}(x)$ lies in the second quadrant, find $\cos x$ and $\tan x$.

9.5.5 Problem Let $x < -1$. Find $\sin \operatorname{arcsec} x$ as a function of x.

9.5.6 Problem Find $\cos \arctan(-\frac{1}{3})$.

9.5.7 Problem Find $\arctan(\tan(-6))$, $\operatorname{arccot}(\cot(-10))$.

9.5.8 Problem Give a sensible definition of the Principal Cotangent, Secant, and Cosecant functions, and their inverses. Graph each of these functions.

9.5.9 Problem Solve the following equations.

1. $\sec^2 x - \sec x - 2 = 0$
2. $\tan x + \cot x = 2$
3. $\tan 4x = 1$
4. $2\sec^2 x + \tan^2 x - 3 = 0$
5. $2\cos x - \sin x = 0$
6. $\tan(x + \frac{\pi}{3}) = 1$
7. $3\cot^2 x + 5\csc x + 1 = 0$
8. $2\sec^2 x = 5\tan x$
9. $\tan^2 x + \sec^2 x = 17$
10. $6\cos^2 x + \sin x - 5 = 0$

9.5.10 Problem Prove that

$$\tan x = \cot\left(\frac{\pi}{2} - x\right),$$

$$\cot x = \tan\left(\frac{\pi}{2} - x\right).$$

9.5.11 Problem Prove that if $x \in \mathbb{R}$ then

$$\arctan x + \operatorname{arccot} \frac{1}{x} = \frac{\pi}{2}\operatorname{sgn}(x),$$

where $\operatorname{sgn}(x) = -1$ if $x < 0$, $\operatorname{sgn}(x) = 1$ if $x > 0$, and $\operatorname{sgn}(0) = 0$.

9.5.12 Problem Graph $x \mapsto (\arctan \circ \tan)(x)$

9.5.13 Problem Let $x \in]0;1[$. Prove that

$$\arcsin x = \operatorname{arccot} \frac{\sqrt{1-x^2}}{x}.$$

9.5.14 Problem Let $x \in]0;1[$. Prove that

$$\arccos x = \arctan \frac{\sqrt{1-x^2}}{x} = \operatorname{arccot} \frac{x}{\sqrt{1-x^2}}.$$

9.5.15 Problem Let $x > 0$. Prove that

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}} = \arccos \frac{1}{\sqrt{1+x^2}}.$$

9.5.16 Problem Let $x > 0$. Prove that

$$\operatorname{arccot} x = \arcsin \frac{1}{\sqrt{1+x^2}} = \arccos \frac{x}{\sqrt{1+x^2}}.$$

9.5.17 Problem Prove the following identities. Assume, whenever necessary, that the given expressions are defined.

1. $$\sin x \tan x = \sec x - \cos x$$
2. $\tan^3 x + 1 = (\tan x + 1)(\sec^2 x - \tan x)$
3. $1 + \tan^2 x = \dfrac{1}{2 - 2\sin x} + \dfrac{1}{2 + 2\sin x}$
4. $\dfrac{\sec\alpha \sin\alpha}{\tan\alpha + \cot\alpha} = \sin^2\alpha$
5. $\dfrac{1 - \sin\alpha}{\cos\alpha} = \dfrac{\cos\alpha}{1 + \sin\alpha}$
6. $7\sec^2 x - 6\tan^2 x + 9\cos^2 x = \dfrac{(1 + 3\cos^2 x)^2}{\cos^2 x}$
7. $\dfrac{1 - \tan^2 t}{1 + \tan^2 t} = \cos^2 t - \sin^2 t$
8. $\dfrac{1 + \tan B + \sec B}{1 + \tan B - \sec B} = (1 + \sec B)(1 + \csc B)$

9.6 Addition Formulae

We will now derive the following formulæ.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{9.21}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \tag{9.22}$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \tag{9.23}$$

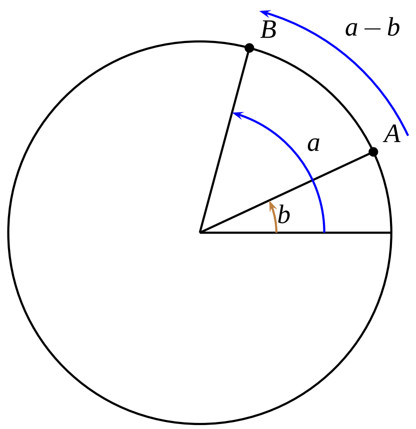


Figure 9.30: Theorem 449.

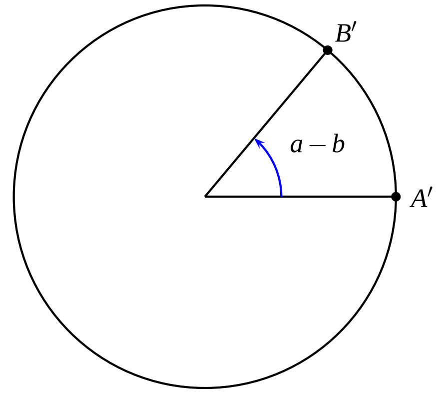


Figure 9.31: Theorem 449.

We begin by proving

449 Theorem Let $(a,b) \in \mathbb{R}^2$. Then $\cos(a-b) = \cos a\cos b + \sin a\sin b$.

Proof: *Consider the points* $A(\cos b, \sin b)$ *and* $B(\cos a, \sin a)$ *in figure 9.30. Their distance is*

$$\begin{aligned}\sqrt{(\cos b - \cos a)^2 + (\sin b - \sin a)^2} &= \sqrt{\cos^2 b - 2\cos b\cos a + \cos^2 a + \sin^2 b - 2\sin b\sin a + \sin^2 a}\\ &= \sqrt{2 - 2(\cos a\cos b + \sin a\sin b)}.\end{aligned}$$

If we rotate A b radians clockwise to $A'(1,0)$*, and B b radians clockwise to* $B'(\cos(a-b), \sin(a-b))$ *as in figure 9.31, the distance is preserved, that is, the distance of* A' *to* B'*, which is*

$$\sqrt{(\cos(a-b)-1) + \sin^2(a-b)} = \sqrt{1 - 2\cos(a-b) + \cos^2(a-b) + \sin^2(a-b)} = \sqrt{2 - 2\cos(a-b)},$$

then equals the distance of A to B. Therefore we have

$$\begin{aligned}\sqrt{2 - 2(\cos a\cos b + \sin a\sin b)} = \sqrt{2 - 2\cos(a-b)} &\implies 2 - 2(\cos a\cos b + \sin a\sin b) = 2 - 2\cos(a-b)\\ &\implies \cos(a-b) = \cos a\cos b + \sin a\sin b.\end{aligned}$$

❑

450 Corollary $\cos(a+b) = \cos a\cos b - \sin a\sin b$.

Proof: *This follows by replacing b by* $-b$ *in Theorem 449, using the fact that* $x \mapsto \cos x$ *is an even function and so* $\cos(-b) = \cos b$*, and that* $x \mapsto \sin x$ *is an odd function and so* $\sin(-b) = -\sin b$*:*

$$\cos(a+b) = \cos(a-(-b)) = \cos a\cos(-b) + \sin a\sin(-b) = \cos a\cos b - \sin a\sin b.$$

❑

451 Theorem Let $(a,b) \in \mathbb{R}^2$. Then $\sin(a \pm b) = \sin a\cos b \pm \sin b\cos a$.

Proof: *We use the fact that* $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ *and that* $\cos x = \sin\left(\frac{\pi}{2} - x\right)$. *Thus*

$$\begin{aligned}\sin(a+b) &= \cos\left(\frac{\pi}{2} - (a+b)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - a\right) - b\right)\\ &= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b\\ &= \sin a\cos b + \cos a\sin b,\end{aligned}$$

proving the addition formula. For the difference formula, we have

$$\sin(a-b) = \sin(a+(-b)) = \sin a\cos(-b) + \sin(-b)\cos a = \sin a\cos b - \sin b\cos a.$$

❑

452 Theorem Let $(a,b) \in \mathbb{R}^2$. Then $\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a\tan b}$.

Proof: *Using the formulæ derived above,*

$$\begin{aligned}\tan(a \pm b) &= \frac{\sin(a \pm b)}{\cos(a \pm b)}\\ &= \frac{\sin a\cos b \pm \sin b\cos a}{\cos a\cos b \mp \sin a\sin b}.\end{aligned}$$

Dividing numerator and denominator by $\cos a\cos b$ *we obtain the result.* ❑

By letting $a+b = A, a-b = B$ in the above results we obtain the following corollary.

453 Corollary

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{9.24}$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{9.25}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{9.26}$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \tag{9.27}$$

454 Example Given that $\cos a = -.1$ and $\pi < a < \frac{3\pi}{2}$, and that $\sin b = .2$ and $0 < b < \frac{\pi}{2}$, find $\cos(a+b)$.

Solution: ► *Since* $\mathscr{C}(a)$ *is in the third quadrant,* $\sin a = -\sqrt{1-(.1)^2} = -\sqrt{0.99}$. *As* $\mathscr{C}(b)$ *is in the first quadrant,* $\cos b = \sqrt{1-(.2)^2} = \sqrt{0.96}$. *By the addition formula for the cosine*

$$\begin{aligned}\cos(a+b) &= \cos a\cos b - \sin a\sin b\\ &= (-.1)(\sqrt{0.96}) - (-\sqrt{0.99})(.2)\\ &= .2\sqrt{.99} - .1\sqrt{.96}.\end{aligned}$$

◄

455 Example Write $\sin 5x\cos x$ as a sum of sines.

Solution: ► *We have*

$$\sin 6x = \sin(5x+x) = \sin 5x\cos x + \sin x\cos 5x$$

$$\sin 4x = \sin(5x-x) = \sin 5x\cos x - \sin x\cos 5x$$

Adding both equalities and dividing by 2*, we gather,*

$$\sin 5x\cos x = \frac{1}{2}\sin 6x + \frac{1}{2}\sin 4x.$$

◄

456 Example Solve the equation

$$\sin 6x + \sin 4x = 0.$$

Solution: ► *As* $\sin 6x + \sin 4x = 2\sin 5x\cos x$ *we must have either* $\sin 5x = 0$ *or* $\cos x = 0$*. Thus*

$$x = \frac{\pi n}{5},\quad x = \pm\frac{\pi}{2} + \pi n, n \in \mathbb{Z}.$$

◄

457 Example Write $\sin x\sin 2x$ as a sum of cosines.

Solution: ► *We have*

$$\cos 3x = \cos(2x+x) = \cos 2x\cos x - \sin 2x\sin x,$$

$$\cos x = \cos(2x-x) = \cos 2x\cos x + \sin 2x\sin x.$$

Subtracting both equalities $\cos 3x - \cos x = -2\sin 2x\sin x$*, whence*

$$\sin 2x\sin x = -\frac{1}{2}\cos 3x + \frac{1}{2}\cos x.$$

◄

458 Example Find the exact value of $\cos\frac{7\pi}{12}$.

Solution: ► *Observe that* $\frac{7}{12} = \frac{1}{3} + \frac{1}{4}$*. Using the addition formulæ*

$$\begin{array}{rcl}\cos\frac{7\pi}{12} & = & \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ & = & \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ & = & (\frac{1}{2})(\frac{\sqrt{2}}{2}) - (\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2})\\ & = & \frac{\sqrt{2}-\sqrt{6}}{4}\end{array}.$$

◄

459 Example (i) Write $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x-\theta)$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. (ii) Use the preceding identity in order to solve the equation

$$\sqrt{3}\cos x + \sin x = -1.$$

(iii) Find all the solutions in the interval $[0;2\pi]$.

Solution: ► *First observe that* $A \neq 0$, *since* $\sqrt{3}\cos x + \sin x$ *is not identically* 0. *We have*

$$A\cos(x-\theta) = A\cos x\cos\theta + A\sin x\sin\theta.$$

If the expression on the dextral side of the above equality is to be equal to $\sqrt{3}\cos x + \sin x$ *then* $A\cos\theta = \sqrt{3}$ *and* $A\sin\theta = 1$. *This entails that* $\tan\theta = \frac{\sqrt{3}}{3}$ *and so* $\theta = \frac{\pi}{6}$. *This in turn yields* $A = 2$. *Hence*

$$\sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right).$$

Now, if $2\cos\left(x - \frac{\pi}{6}\right) = -1$, *then*

$$x - \frac{\pi}{6} = \pm\arccos(-\frac{1}{2}) + 2n\pi,\ n \in \mathbb{Z},$$

$$x = \frac{\pi}{6} \pm \frac{2\pi}{3} + 2n\pi,\ n \in \mathbb{Z},$$

which is the same family as $x = \frac{5\pi}{6} + 2n\pi, x = -\frac{\pi}{2} + 2n\pi$ *and the solutions in* $[0;2\pi]$ *are clearly* $x = \frac{5\pi}{6}$ *and* $x = \frac{3\pi}{2}$.

Aliter: *Write the equation as* $\sqrt{3}\cos x + 1 = -\sin x$ *and square*

$$3\cos^2 x + 2\sqrt{3}\cos x + 1 = \sin^2 x.$$

Using $\sin^2 x = 1 - \cos^2 x$ *we obtain*

$$3\cos^2 x + 2\sqrt{3}\cos x + 1 = 1 - \cos^2 x,$$

or

$$(\cos x)(4\cos x + 2\sqrt{3}) = 0.$$

This equation has solutions $x = \pm\frac{\pi}{2} + 2n\pi$ *and* $x = \pm\frac{5\pi}{6} + 2n\pi$. *Testing* $x = \frac{\pi}{2}$ *in the original equation* $\sqrt{3}\cos x + \sin x = -1$ *we see that it is not a solution, hence the family* $x = \frac{\pi}{2} + 2n\pi$ *is not part of the solution set of the original equation. The same happens when we test* $x = -\frac{5\pi}{6}$, *so we must also discard this family. The two remaining families,* $x = \frac{5\pi}{6} + 2n\pi$, $x = -\frac{\pi}{2} + 2n\pi$ *agree with our previous solution.* ◄

460 Example Obtain a formula for $\cos(a+b+c)$ in terms of cosines and sines of a, b, and c.

Solution: ► *Using the addition formula twice*

$$\begin{aligned}
\cos(a+b+c) &= \cos a\cos(b+c) - \sin a\sin(b+c)\\
&= \cos a(\cos b\cos c - \sin b\sin c) -\\
&\quad -\sin a(\sin b\cos c + \sin c\cos b)\\
&= \cos a\cos b\cos c - \cos a\sin b\sin c -\\
&\quad -\sin a\sin b\cos c - \sin a\cos b\sin c
\end{aligned}$$

◄

461 Example (Canadian Mathematical Olympiad 1984) Given any 7 real numbers, prove that there are two of them, say, x and y, such that

$$0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$

Solution: ▶ *Let the numbers be $a_k, k = 1, 2, \ldots, 7$. There exists b_k such $a_k = \tan b_k$, since*

$$\begin{array}{ccc}]-\frac{\pi}{2};\frac{\pi}{2}[& \rightarrow & \mathbb{R} \\ x & \mapsto & \tan x \end{array}$$

is a bijection. Divide the interval $]-\frac{\pi}{2};\frac{\pi}{2}[$ into six subintervals, each of length $\frac{\pi}{6}$. Since we have 7 b_k's and 6 subintervals, two of the b_k's, say b_s and b_t, must lie in the same subinterval. Assuming $b_s \geq b_t$ we then have $0 \leq b_s - b_t \leq \frac{\pi}{6}$. Since $x \mapsto \tan x$ is an increasing function,

$$\tan 0 \leq \tan(b_s - b_t) \leq \tan\frac{\pi}{6},$$

which is to say,

$$0 \leq \frac{\tan b_s - \tan b_t}{1 + \tan b_s \tan b_t} \leq \frac{1}{\sqrt{3}}.$$

This implies that

$$0 \leq \frac{a_s - a_t}{1 + a_s a_t} \leq \frac{1}{\sqrt{3}},$$

which completes the proof. ◀

462 Example Prove that if

$$\frac{a-b}{1+ab} + \frac{b-c}{1+bc} + \frac{c-a}{1+ca} = 0,$$

for real numbers a, b, c, then at least two of the numbers a, b, c are equal.

Solution: ▶ *$\exists u, v, w$ with $-\frac{\pi}{2} < u, v, w < \frac{\pi}{2}$ such that $a = \tan u, b = \tan v, c = \tan w$ (why?). The given equation becomes*

$$\frac{\tan u - \tan v}{1 + \tan u \tan v} + \frac{\tan v - \tan w}{1 + \tan v \tan w} + \frac{\tan w - \tan u}{1 + \tan w \tan u} = 0.$$

Using the addition for the tangents, the preceding relation is equivalent to

$$\tan(u - v) + \tan(v - w) + \tan(w - u) = 0.$$

Applying $\tan X + \tan Y = (\tan(X+Y))(1 - \tan X \tan Y)$ with $X = u - v$ and $Y = v - w$, we obtain

$$(\tan(u - w))(1 - \tan(u - v)\tan(v - w)) + \tan(w - u) = 0.$$

Factorising the above expression,

$$(\tan(u - w))(\tan(u - v))(\tan(v - w)) = 0.$$

This implies that one of the tangents in this product must be 0. Since

$$-\pi < u - w, u - v, v - w < \pi,$$

this means that one of these differences must be exactly 0, which in turn implies that two of the numbers a, b, c are equal. ◀

463 Example Prove that

$$\arctan a + \arctan b = \begin{cases} \arctan\frac{a+b}{1-ab} & \text{if } ab < 1, \\ \frac{\pi}{2}(\mathrm{sgn}(a)) & \text{if } ab = 1, \\ \arctan\frac{a+b}{1-ab} + \frac{\pi}{2}(\mathrm{sgn}(a)) & \text{if } ab > 1. \end{cases}$$

Solution: ▶ *Put $x = \arctan a, y = \arctan b$. If $(x, y) \in]-\frac{\pi}{2};\frac{\pi}{2}[^2$ and $x + y \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$, then*

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{a + b}{1 - ab}.$$

Now, $-\pi < x+y < \pi$. *Conditioning on* x *we have,*

$$-\frac{\pi}{2} < x+y < \frac{\pi}{2} \iff \left| \begin{array}{l} x = 0 \\ \text{or } x > 0 \text{ and } y < \frac{\pi}{2} - x \\ \text{or } x < 0 \text{ and } y > -\frac{\pi}{2} - x \end{array} \right.$$

The above choices hold if and only if

$$\begin{array}{l} a = 0 \\ \text{or } a > 0 \text{ and } b < \frac{1}{a} \\ \text{or } a < 0 \text{ and } b > \frac{1}{a} \end{array} .$$

Hence, if $ab < 1$, *then* $x+y \in]-\frac{\pi}{2};\frac{\pi}{2}[$ *and thus*

$$x+y = \arctan(\tan(x+y)) = \arctan\frac{a+b}{1-ab}.$$

If $ab > 1$ *and* $a > 0$ *then* $x+y \in]\frac{\pi}{2};\pi[$ *and thus*

$$x+y = \arctan\frac{a+b}{1-ab} + \pi.$$

If $ab > 1$ *and* $a < 0$, *then* $x+y \in]-\pi;-\frac{\pi}{2}[$ *and thus*

$$x+y = \arctan\frac{a+b}{1-ab} - \pi.$$

The case $ab = 1$ *is left as an exercise.* ◀

464 Example Solve the equation $\arccos x = \arcsin\frac{1}{3} + \arccos\frac{1}{4}$.

Solution: ▶ *Observe that* $\arccos x \in [0;\pi]$ *and that since both* $0 \le \arcsin\frac{1}{3} \le \frac{\pi}{2}$ *and* $0 \le \arccos\frac{1}{4} \le \frac{\pi}{2}$, *we have* $0 \le \arcsin\frac{1}{3} + \arccos\frac{1}{4} \le \pi$. *Hence, we may take cosines on both sides of the equation and obtain*

$$\begin{array}{rcl} x & = & \cos(\arccos x) \\ & = & \cos(\arcsin\frac{1}{3} + \arccos\frac{1}{4}) \\ & = & (\cos\arcsin\frac{1}{3})(\cos\arccos\frac{1}{4}) - (\sin\arcsin\frac{1}{3})(\sin\arccos\frac{1}{4}) \\ & = & \frac{\sqrt{2}}{6} - \frac{\sqrt{15}}{12} \end{array} .$$

◀

465 Example (Machin's Formula) Prove that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

Solution: ► *Observe that*

$$\begin{aligned}
4\arctan\tfrac{1}{5} &= 2\arctan\tfrac{1}{5}+2\arctan\tfrac{1}{5}\\
&= 2\arctan\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}}\\
&= 2\arctan\tfrac{5}{12}\\
&= \arctan\tfrac{5}{12}+\arctan\tfrac{5}{12}\\
&= \arctan\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{5}{12}\cdot\frac{5}{12}}\\
&= \arctan\tfrac{120}{119}.
\end{aligned}$$

Also

$$\begin{aligned}
\arctan\tfrac{120}{119}-\arctan\tfrac{1}{239} &= \arctan\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}\\
&= \arctan 1\\
&= \tfrac{\pi}{4}.
\end{aligned}$$

Upon assembling the equalities, we obtain the result. ◄

Homework

9.6.1 Problem Demonstrate the identity

$$\sin(a+b)\sin(a-b)=\sin^2 a-\sin^2 b=\cos^2 b-\cos^2 a$$

9.6.2 Problem Prove that for all real numbers x,

$$\cos\left(2x-\frac{4\pi}{3}\right)+\cos 2x+\cos\left(2x+\frac{4\pi}{3}\right)=0.$$

9.6.3 Problem Using the fact that $\frac{1}{12}=\frac{1}{3}-\frac{1}{4}$, find the exact value of the following.

1. $\cos\pi/12$
2. $\sin\pi/12$

9.6.4 Problem Write $\cot(a+b)$ in terms of $\cot a$ and $\cot b$.

9.6.5 Problem Write $\sin x\sin 2x$ as a sum of cosines.

9.6.6 Problem Write $\cos x\cos 4x$ as a sum of cosines.

9.6.7 Problem Write using only one arcsine: $\arccos\frac{4}{5}-\arccos\frac{1}{4}$.

9.6.8 Problem Write using only one arctangent: $\arctan\frac{1}{3}-\arctan\frac{1}{4}$.

9.6.9 Problem Write using only one arctangent: $\text{arccot}\,(-2)-\arctan(-\frac{2}{3})$.

9.6.10 Problem Write $\sin x\cos 2x$ as a sum of sines.

9.6.11 Problem Write $\sin x\sin 2x\sin 3x$ as a sum of sines.

9.6.12 Problem Given real numbers a,b with $0<a<\pi/2$ and $\pi<b<3\pi/2$ and given that $\sin a=1/3$ and $\cos b=-1/2$, find $\cos(a-b)$.

9.6.13 Problem Solve the equation $\cos x+\cos 3x=0.$.

9.6.14 Problem Solve the equation

$$\arcsin(\tan x)=x.$$

9.6.15 Problem Solve the equation

$$\arccos x=\arcsin(1-x).$$

9.6.16 Problem Solve the equation

$$\arctan x+\arctan 2x=\frac{\pi}{4}.$$

9.6.17 Problem Prove the identity

$$\cos^4 x=\frac{1}{8}(\cos 4x+4\cos 2x+3).$$

9.6.18 Problem Prove the identities

$$\tan a + \tan b = \frac{\sin(a+b)}{(\cos a)(\cos b)},$$

$$\cot a + \cot b = \frac{\sin(a+b)}{(\sin a)(\sin b)}.$$

9.6.19 Problem Given that $0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{2}$ and satisfy $\sin\alpha = 12/13, \cos\beta = 8/17, \sin\gamma = 4/5$, find the value of $\sin(\alpha+\beta-\gamma)$ and $\cos(\alpha-\beta+2\gamma)$.

9.6.20 Problem Establish the identity

$$\frac{\sin(a-b)\sin(a+b)}{1-\tan^2 a\cot^2 b} = -\cos^2 a\sin^2 b.$$

9.6.21 Problem Find real constants a, b, c such that

$$\sin 3x - \sqrt{3}\cos 3x = a\sin(bx+c).$$

Use this to solve the equation

$$\sin 3x - \sqrt{3}\cos 3x = -\sqrt{2}.$$

9.6.22 Problem Solve the equation

$$\sin 2x + \cos 2x = -1$$

9.6.23 Problem Simplify: $\sin(\operatorname{arcsec}\dfrac{17}{8} - \arctan(-\dfrac{2}{3}))$.

9.6.24 Problem Shew that if $\cot(a+b) = 0$ then $\sin(a+2b) = \sin a$.

9.6.25 Problem Let $a+b+c = \frac{\pi}{2}$. Write $\cos a\cos b\cos c$ as a sum of sines.

9.6.26 Problem Shew that the amplitude of $x \mapsto a\sin Ax + b\cos Ax$ is $\sqrt{a^2+b^2}$.

9.6.27 Problem Solve the equation

$$\cos x - \sin x = 1.$$

9.6.28 Problem Let $a+b+c = \pi$. Simplify

$$\sin^2 a + \sin^2 b + \sin^2 c - 2\cos a\cos b\cos c.$$

9.6.29 Problem Prove that if

$$\cot a + \csc a\cos b\sec c = \cot b + \cos a\csc b\sec c,$$

then either $a-b = k\pi$, or $a+b+c = \pi+2m\pi$ or $a+b-c = \pi+2n\pi$ for some integers k, m, n.

9.6.30 Problem Prove that if

$$\tan a + \tan b + \tan c = \tan a\tan b\tan c,$$

then $a+b+c = k\pi$ for some integer k.

9.6.31 Problem Prove that if any of $a+b+c$, $a+b-c$, $a-b+c$ or $a-b-c$ is equal to an odd multiple of π, then

$$\cos^2 a + \cos^2 b + \cos^2 c + 2\cos a\cos b\cos c = 1,$$

and that the converse is also true.

A Complex Numbers

A.1 Arithmetic of Complex Numbers

One uses the symbol i to denote the *imaginary unit* $i = \sqrt{-1}$. Then $i^2 = -1$.

466 Example Find $\sqrt{-25}$.

Solution: ► $\sqrt{-25} = 5i$. ◄

Since $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$, etc., the powers of i repeat themselves cyclically in a cycle of period 4.

467 Example Find i^{1934}.

Solution: ► *Observe that* $1934 = 4(483)+2$ *and so* $i^{1934} = i^2 = -1$. ◄

468 Example For any integral α one has

$$i^{\alpha} + i^{\alpha+1} + i^{\alpha+2} + i^{\alpha+3} = i^{\alpha}(1+i+i^2+i^3) = i^{\alpha}(1+i-1-i) = 0.$$

If a, b are real numbers then the object $a+bi$ is called a *complex number*. One uses the symbol $\mathbb{C}$ to denote the set of all complex numbers. If $a, b, c, d \in \mathbb{R}$, then the sum of the complex numbers $a+bi$ and $c+di$ is naturally defined as

$$(a+bi)+(c+di) = (a+c)+(b+d)i \tag{A.1}$$

The product of $a+bi$ and $c+di$ is obtained by multiplying the binomials:

$$(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i \tag{A.2}$$

469 Example Find the sum $(4+3i)+(5-2i)$ and the product $(4+3i)(5-2i)$.

Solution: ► *One has*

$$(4+3i)+(5-2i) = 9+i$$

and

$$(4+3i)(5-2i) = 20-8i+15i-6i^2 = 20+7i+6 = 26+7i.$$

◄

470 Definition Let $z \in \mathbb{C}, (a,b) \in \mathbb{R}^2$ with $z = a+bi$. The *conjugate* $\overline{z}$ of z is defined by

$$\overline{z} = \overline{a+bi} = a-bi \tag{A.3}$$

471 Example The conjugate of $5+3i$ is $\overline{5+3i} = 5-3i$. The conjugate of $2-4i$ is $\overline{2-4i} = 2+4i$.

The conjugate of a real number is itself, that is, if $a \in \mathbb{R}$, then $\overline{a} = a$. Also, the conjugate of the conjugate of a number is the number, that is, $\overline{\overline{z}} = z$.

472 Theorem The function $z : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$ is multiplicative, that is, if z_1, z_2 are complex numbers, then

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \tag{A.4}$$

Proof: *Let $z_1 = a+bi, z_2 = c+di$ where a,b,c,d are real numbers. Then*

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(a+bi)(c+di)}\\ &= \overline{(ac-bd)+(ad+bc)i}\\ &= (ac-bd)-(ad+bc)i\end{aligned}$$

Also,

$$\begin{aligned}\overline{z_1}\cdot\overline{z_2} &= (\overline{a+bi})(\overline{c+di})\\ &= (a-bi)(c-di)\\ &= ac-adi-bci+bdi^2\\ &= (ac-bd)-(ad+bc)i,\end{aligned}$$

which establishes the equality between the two quantities. ❑

473 Example Express the quotient $\dfrac{2+3i}{3-5i}$ in the form $a+bi$.

Solution: ► *One has*

$$\frac{2+3i}{3-5i} = \frac{2+3i}{3-5i}\cdot\frac{3+5i}{3+5i} = \frac{-9+19i}{34} = \frac{-9}{34}+\frac{19i}{34}$$

◄

474 Definition The *modulus* $|a+bi|$ of $a+bi$ is defined by

$$|a+bi| = \sqrt{(a+bi)(\overline{a+bi})} = \sqrt{a^2+b^2} \tag{A.5}$$

Observe that $z \mapsto |z|$ is a function mapping $\mathbb{C}$ to $[0;+\infty[$.

475 Example Find $|7+3i|$.

Solution: ► $|7+3i| = \sqrt{(7+3i)(7-3i)} = \sqrt{7^2+3^2} = \sqrt{58}$. ◄

476 Example Find $|\sqrt{7}+3i|$.

Solution: ► $|\sqrt{7}+3i| = \sqrt{(\sqrt{7}+3i)(\sqrt{7}-3i)} = \sqrt{7+3^2} = 4$. ◄

477 Theorem The function $z \mapsto |z|$, $\mathbb{C} \to \mathbb{R}_+$ is multiplicative. That is, if z_1, z_2 are complex numbers then

$$|z_1 z_2| = |z_1||z_2| \tag{A.6}$$

Proof: *By Theorem 472, conjugation is multiplicative, hence*

$$\begin{aligned}|z_1z_2| &= \sqrt{z_1z_2\overline{z_1z_2}} \\ &= \sqrt{z_1z_2\overline{z_1}\cdot\overline{z_2}} \\ &= \sqrt{z_1\overline{z_1}z_2\overline{z_2}} \\ &= \sqrt{z_1\overline{z_1}}\sqrt{z_2\overline{z_2}} \\ &= |z_1||z_2|\end{aligned}$$

whence the assertion follows. ❑

478 Example Write $(2^2+3^2)(5^2+7^2)$ as the sum of two squares.

Solution: ► *The idea is to write* $2^2+3^2 = |2+3i|^2$, $5^2+7^2 = |5+7i|^2$ *and use the multiplicativity of the modulus. Now*

$$\begin{aligned}(2^2+3^2)(5^2+7^2) &= |2+3i|^2|5+7i|^2 \\ &= |(2+3i)(5+7i)|^2 \\ &= |-11+29i|^2 \\ &= 11^2+29^2\end{aligned}$$

◄

A.2 Equations involving Complex Numbers

Recall that if $ux^2+vx+w=0$ with $u\neq 0$, then the roots of this equation are given by the *Quadratic Formula*

$$x = -\frac{v}{2u} \pm \frac{\sqrt{v^2-4uw}}{2u} \tag{A.7}$$

The quantity v^2-4uw under the square root is called the *discriminant* of the quadratic equation $ux^2+vx+w=0$. If u,v,w are real numbers and this discriminant is negative, one obtains complex roots.
Complex numbers thus occur naturally in the solution of quadratic equations. Since $i^2=-1$, one sees that $x=i$ is a root of the equation $x^2+1=0$. Similary, $x=-i$ is also a root of x^2+1.

479 Example Solve $2x^2+6x+5=0$

Solution: ► *Using the quadratic formula*

$$x = -\frac{6}{4} \pm \frac{\sqrt{-4}}{4} = -\frac{3}{2} \pm i\frac{1}{2}$$

◄

In solving the problems that follow, the student might profit from the following identities.

$$s^2-t^2 = (s-t)(s+t) \tag{A.8}$$

$$s^{2k}-t^{2k} = (s^k-t^k)(s^k+t^k),\ k\in\mathbb{N} \tag{A.9}$$

$$s^3 - t^3 = (s-t)(s^2 + st + t^2) \tag{A.10}$$

$$s^3 + t^3 = (s+t)(s^2 - st + t^2) \tag{A.11}$$

480 Example Solve the equation $x^4 - 16 = 0$.

Solution: ▶ *One has* $x^4 - 16 = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4)$. *Thus either* $x = -2, x = 2$ *or* $x^2 + 4 = 0$. *This last equation has roots* $\pm 2i$. *The four roots of* $x^4 - 16 = 0$ *are thus* $x = -2, x = 2, x = -2i, x = 2i$. ◀

481 Example Find the roots of $x^3 - 1 = 0$.

Solution: ▶ $x^3 - 1 = (x-1)(x^2+x+1)$. *If* $x \neq 1$, *the two solutions to* $x^2 + x + 1 = 0$ *can be obtained using the quadratic formula, getting* $x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. ◀

482 Example Find the roots of $x^3 + 8 = 0$.

Solution: ▶ $x^3 + 8 = (x+2)(x^2 - 2x + 4)$. *Thus either* $x = -2$ *or* $x^2 - 2x + 4 = 0$. *Using the quadratic formula, one sees that the solutions of this last equation are* $x = 1 \pm i\sqrt{3}$. ◀

483 Example Solve the equation $x^4 + 9x^2 + 20 = 0$.

Solution: ▶ *One sees that*

$$x^4 + 9x^2 + 20 = (x^2+4)(x^2+5) = 0$$

Thus either $x^2 + 4 = 0$, *in which case* $x = \pm 2i$ *or* $x^2 + 5 = 0$ *in which case* $x = \pm i\sqrt{5}$. *The four roots are* $x = \pm 2i, \pm i\sqrt{5}$ ◀

Homework

A.2.1 Problem Perform the following operations. Write your result in the form $a + bi$, with $(a, b) \in \mathbb{R}^2$.

1. $\sqrt{36} + \sqrt{-36}$
2. $(4+8i) - (9-3i) + 5(2+i) - 8i$
3. $4 + 5i + 6i^2 + 7i^3$
4. $i(1+i) + 2i^2(3-4i)$
5. $(8-9i)(10+11i)$
6. $i^{1990} + i^{1991} + i^{1992} + i^{1993}$
7. $\dfrac{2-i}{2+i}$
8. $\dfrac{1-i}{1+2i} + \dfrac{1+i}{1+2i}$
9. $(5+2i)^2 + (5-2i)^2$
10. $(1+i)^3$

A.2.2 Problem Find real numbers a, b such that

$$(a-2) + (5b+3)i = 4 - 2i$$

A.2.3 Problem Write $(2^2 + 3^2)(3^2 + 7^2)$ as the sum of two squares.

A.2.4 Problem Prove that $(1+i)^2 = 2i$ and that $(1-i)^2 = -2i$. Use this to write

$$\frac{(1+i)^{2004}}{(1-i)^{2000}}$$

in the form $a + bi$, $(a, b) \in \mathbb{R}^2$.

A.2.5 Problem Prove that $(1 + i\sqrt{3})^3 = 8$. Use this to prove that

$$(1 + i\sqrt{3})^{30} = 2^{30}.$$

A.2.6 Problem Find $|5 + 7i|$, $|\sqrt{5} + 7i|$, $|5 + i\sqrt{7}|$ and $|\sqrt{5} + i\sqrt{7}|$.

A.2.7 Problem Prove that if k is an integer then

$$(4k+1)i^{4k} + (4k+2)i^{4k+1} + (4k+3)i^{4k+2} + (4k+4)i^{4k+3} = -2 - 2i.$$

Use this to prove that

$$1 + 2i + 3i^2 + 4i^3 + \cdots + 1995i^{1994} + 1996i^{1995} = -998 - 998i.$$

A.2.8 Problem If z and z' are complex numbers with either $|z| = 1$ or $|z'| = 1$, prove that

$$\left| \frac{z - z'}{1 - \overline{z}z'} \right| = 1.$$

A.2.9 Problem Prove that if z, z' and w are complex numbers with $|z| = |z'| = |w| = 1$, then

$$|zz' + zw + z'w| = |z + z' + w|$$

A.2.10 Problem Prove that if n is an integer which is not a multiple of 4 then

$$1^n + i^n + i^{2n} + i^{3n} = 0.$$

Now let

$$f(x) = (1 + x + x^2)^{1000} = a_0 + a_1 x + \cdots + a_{2000} x^{2000}.$$

By considering $f(1) + f(i) + f(i^2) + f(i^3)$, find

$$a_0 + a_4 + a_8 + \cdots + a_{2000}.$$

A.2.11 Problem Find all the roots of the following equations.

1. $x^2 + 8 = 0$
2. $x^2 + 49 = 0$
3. $x^2 - 4x + 5 = 0$
4. $x^2 - 3x + 6 = 0$
5. $x^4 - 1 = 0$
6. $x^4 + 2x^2 - 3 = 0$
7. $x^3 - 27 = 0$
8. $x^6 - 1 = 0$
9. $x^6 - 64 = 0$

A.3 Polar Form of Complex Numbers

Complex numbers can be given a geometric representation in the *Argand diagram* (see figure A.1), where the horizontal axis carries the real parts and the vertical axis the imaginary ones.

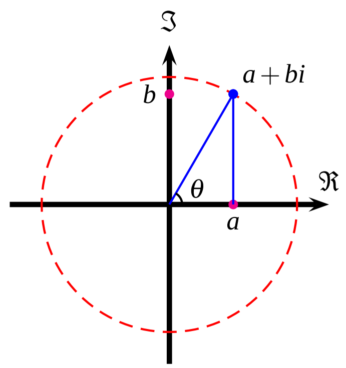


Figure A.1: Argand's diagram.

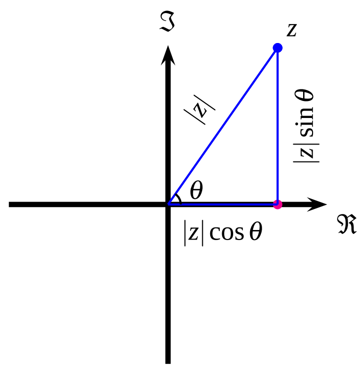


Figure A.2: Polar Form of a Complex Number.

Given a complex number $z = a + bi$ on the Argand diagram, consider the angle $\theta \in]-\pi;\pi]$ that a straight line segment passing through the origin and through z makes with the positive real axis. Considering the polar coordinates of z we gather

$$z = |z|(\cos\theta + i\sin\theta), \qquad \theta \in]-\pi;\pi], \tag{A.12}$$

which we call the *polar form* of the complex number z. The angle θ is called the *argument* of the complex number z.

484 Example Find the polar form of $\sqrt{3} - i$.

Solution: ► *First observe that* $|\sqrt{3} - i| = \sqrt{\sqrt{3}^2 + 1^2} = 2$. *Now, if*

$$\sqrt{3} - i = 2(\cos\theta + i\sin\theta),$$

we need $\cos\theta = \dfrac{\sqrt{3}}{2}$, $\sin\theta = -\dfrac{1}{2}$. *This happens for* $\theta \in]-\pi;\pi]$ *when* $\theta = -\dfrac{\pi}{6}$. *Therefore,*

$$\sqrt{3} - i = 2(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$$

is the required polar form. ◄

We now present some identities involving complex numbers. Let us start with the following classic result. The proof requires Calculus and can be omitted.
If we allow complex numbers in our MacLaurin expansions, we readily obtain Euler's Formula.

485 Theorem (Euler's Formula) Let $x \in \mathbb{R}$. Then

$$e^{ix} = \cos x + i \sin x.$$

Proof: *Using the MacLaurin expansion's of $x \mapsto e^x$, $x \mapsto \cos x$, and $x \mapsto \sin x$, we gather*

$$\begin{aligned} e^{ix} &= \sum_{k=0}^{+\infty} \frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{+\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{k=0}^{+\infty} \frac{(ix)^{2n+1}}{(2n+1)!} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{k=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \cos x + i \sin x. \end{aligned}$$

❑

Taking complex conjugates,

$$e^{-ix} = \overline{e^{ix}} = \overline{\cos x + i \sin x} = \cos x - i \sin x.$$

Solving for $\sin x$ we obtain

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{A.13}$$

Similarly,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{A.14}$$

486 Corollary (De Moivre's Theorem) Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

Proof: *We have*

$$(\cos x + i \sin x)^n = (e^{ix})^n = e^{ixn} = \cos nx + i \sin nx,$$

by theorem 485.

Aliter: *An alternative proof without appealing to Euler's identity follows. We first assume that $n > 0$ and give a proof by induction. For $n = 1$ the assertion is obvious, as*

$$(\cos x + i \sin x)^1 = \cos 1 \cdot x + i \sin 1 \cdot x.$$

Assume the assertion is true for $n - 1 > 1$, that is, assume that

$$(\cos x + i \sin x)^{n-1} = \cos (n-1)x + i \sin (n-1)x.$$

Using the addition identities for the sine and cosine,

$$\begin{aligned} (\cos x + i \sin x)^n &= (\cos x + i \sin x)(\cos x + i \sin x)^{n-1} \\ &= (\cos x + i \sin x)(\cos (n-1)x + i \sin (n-1)x). \\ &= (\cos x)(\cos (n-1)x) - (\sin x)(\sin (n-1)x) + i((\cos x)(\sin (n-1)x) + (\cos (n-1)x)(\sin x)). \\ &= \cos (n-1+1)x + i \sin (n-1+1)x \\ &= \cos nx + i \sin nx, \end{aligned}$$

proving the theorem for $n > 0$.

Assume now that $n < 0$. Then $-n > 0$ and we may used what we just have proved for positive integers we have

$$
\begin{aligned}
(\cos x + i\sin x)^n &= \frac{1}{(\cos x + i\sin x)^{-n}} \\
&= \frac{1}{\cos(-nx) + i\sin(-nx)} \\
&= \frac{1}{\cos nx - i\sin nx} \\
&= \frac{\cos nx + i\sin nx}{(\cos nx + i\sin nx)(\cos nx - i\sin nx)} \\
&= \frac{\cos nx + i\sin nx}{\cos^2 nx + \sin^2 nx} \\
&= \cos nx + i\sin nx,
\end{aligned}
$$

proving the theorem for $n < 0$. If $n = 0$, then since sin *and* cos *are not simultaneously zero, we get* $1 = (\cos x + i\sin x)^0 = \cos 0x + i\sin 0x = \cos 0x = 1$, *proving the theorem for $n = 0$.*

❑

487 Example Prove that

$$\cos 3x = 4\cos^3 x - 3\cos x, \qquad \sin 3x = 3\sin x - 4\sin^3 x.$$

Solution: ► *Using Euler's identity and the Binomial Theorem,*

$$
\begin{aligned}
\cos 3x + i\sin 3x &= e^{3ix} \\
&= (e^{ix})^3 = (\cos x + i\sin x)^3 \\
&= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x \\
&= \cos^3 x + 3i(1-\sin^2 x)\sin x - 3\cos x(1-\cos^2 x) - i\sin^3 x,
\end{aligned}
$$

we gather the required identities. ◄

The following corollary is immediate.

488 Corollary (Roots of Unity) If $n > 0$ is an integer, the n numbers $e^{2\pi ik/n} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, $0 \le k \le n-1$, are all different and satisfy $(e^{2\pi ik/n})^n = 1$.

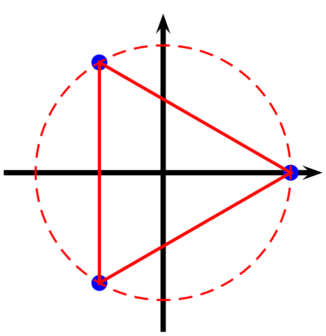


Figure A.3: Cubic Roots of 1.

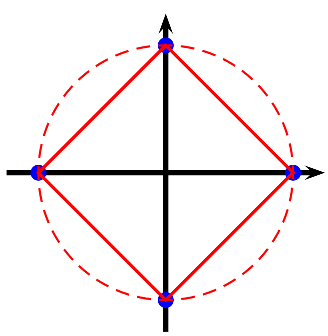


Figure A.4: Quartic Roots of 1.

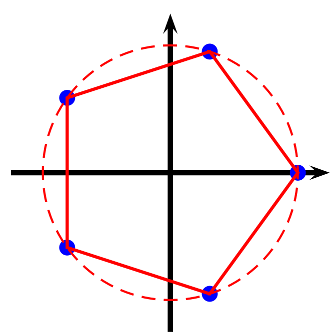


Figure A.5: Quintic Roots of 1.

489 Example For $n = 2$, the square roots of unity are the roots of

$$x^2 - 1 = 0 \implies x \in \{-1, 1\}.$$

For $n = 3$ we have $x^3 - 1 = (x-1)(x^2+x+1) = 0$ hence if $x \neq 1$ then $x^2+x+1 = 0 \implies x = \dfrac{-1 \pm i\sqrt{3}}{2}$. Hence the cubic roots of unity are

$$\left\{-1, \frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\right\}.$$

Or, we may find them trigonometrically,

$$\begin{array}{rcccl} e^{2\pi i\cdot 0/3} & = & \cos\dfrac{2\pi\cdot 0}{3} + i\sin\dfrac{2\pi\cdot 0}{3} & = & 1, \\ e^{2\pi i\cdot 1/3} & = & \cos\dfrac{2\pi\cdot 1}{3} + i\sin\dfrac{2\pi\cdot 1}{3} & = & -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} \\ e^{2\pi i\cdot 2/3} & = & \cos\dfrac{2\pi\cdot 2}{3} + i\sin\dfrac{2\pi\cdot 2}{3} & = & -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2} \end{array}$$

For $n = 4$ they are the roots of $x^4 - 1 = (x-1)(x^3+x^2+x+1) = (x-1)(x+1)(x^2+1) = 0$, which are clearly

$$\{-1, 1, -i, i\}.$$

Or, we may find them trigonometrically,

$$\begin{array}{rcccl} e^{2\pi i\cdot 0/4} & = & \cos\dfrac{2\pi\cdot 0}{4} + i\sin\dfrac{2\pi\cdot 0}{4} & = & 1, \\ e^{2\pi i\cdot 1/4} & = & \cos\dfrac{2\pi\cdot 1}{4} + i\sin\dfrac{2\pi\cdot 1}{4} & = & i \\ e^{2\pi i\cdot 2/4} & = & \cos\dfrac{2\pi\cdot 2}{4} + i\sin\dfrac{2\pi\cdot 2}{4} & = & -1 \\ e^{2\pi i\cdot 3/4} & = & \cos\dfrac{2\pi\cdot 3}{4} + i\sin\dfrac{2\pi\cdot 3}{4} & = & -i \end{array}$$

For $n = 5$ they are the roots of $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1) = 0$. To solve $x^4+x^3+x^2+x+1 = 0$ observe that since clearly $x \neq 0$, by dividing through by x^2, we can transform the equation into

$$x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0.$$

Put now $u = x + \dfrac{1}{x}$. Then $u^2 - 2 = x^2 + \dfrac{1}{x^2}$, and so

$$x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0 \implies u^2 - 2 + u + 1 = 0 \implies u = \frac{-1 \pm \sqrt{5}}{2}.$$

Solving both equations

$$x + \frac{1}{x} = \frac{-1-\sqrt{5}}{2}, \qquad x + \frac{1}{x} = \frac{-1+\sqrt{5}}{2},$$

we get the four roots

$$\left\{\frac{-1-\sqrt{5}}{4} - i\frac{\sqrt{10-2\sqrt{5}}}{4},\quad \frac{-1-\sqrt{5}}{4} + i\frac{\sqrt{10-2\sqrt{5}}}{4},\quad \frac{\sqrt{5}-1}{4} - i\frac{\sqrt{2\sqrt{5}+10}}{4},\quad \frac{\sqrt{5}-1}{4} + i\frac{\sqrt{2\sqrt{5}+10}}{4}\right\},$$

or, we may find, trigonometrically,

$$\begin{aligned}
e^{2\pi i\cdot 0/5} &= \cos\frac{2\pi\cdot 0}{5}+i\sin\frac{2\pi\cdot 0}{5} &= 1,\\
e^{2\pi i\cdot 1/5} &= \cos\frac{2\pi\cdot 1}{5}+i\sin\frac{2\pi\cdot 1}{5} &= \left(\frac{\sqrt{5}-1}{4}\right)+i\left(\frac{\sqrt{2}\cdot\sqrt{5+\sqrt{5}}}{4}\right),\\
e^{2\pi i\cdot 2/5} &= \cos\frac{2\pi\cdot 2}{5}+i\sin\frac{2\pi\cdot 2}{5} &= \left(\frac{-\sqrt{5}-1}{4}\right)+i\left(\frac{\sqrt{2}\cdot\sqrt{5-\sqrt{5}}}{4}\right),\\
e^{2\pi i\cdot 3/5} &= \cos\frac{2\pi\cdot 3}{5}+i\sin\frac{2\pi\cdot 3}{5} &= \left(\frac{-\sqrt{5}-1}{4}\right)-i\left(\frac{\sqrt{2}\cdot\sqrt{5-\sqrt{5}}}{4}\right),\\
e^{2\pi i\cdot 4/5} &= \cos\frac{2\pi\cdot 4}{5}+i\sin\frac{2\pi\cdot 4}{5} &= \left(\frac{\sqrt{5}-1}{4}\right)-i\left(\frac{\sqrt{2}\cdot\sqrt{5+\sqrt{5}}}{4}\right),
\end{aligned}$$

See figures A.3 through A.5.

By the Fundamental Theorem of Algebra the equation $x^n-1=0$ has exactly n complex roots, which gives the following result.

490 Corollary Let $n>0$ be an integer. Then

$$x^n-1=\prod_{k=0}^{n-1}(x-e^{2\pi ik/n}).$$

491 Theorem We have,

$$1+x+x^2+\cdots+x^{n-1}=\begin{cases}0 & x=e^{\frac{2\pi ik}{n}},\quad 1\leq k\leq n-1,\\ n & x=1.\end{cases}$$

Proof: *Since $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$, from Corollary 490, if $x\neq 1$,*

$$x^{n-1}+x^{n-2}+\cdots+x+1=\prod_{k=1}^{n-1}(x-e^{2\pi ik/n}).$$

If ε is a root of unity different from 1, then $\varepsilon=e^{2\pi ik/n}$ for some $k\in[1;n-1]$, and this proves the theorem. Alternatively,

$$1+\varepsilon+\varepsilon^2+\varepsilon^3+\cdots+\varepsilon^{n-1}=\frac{\varepsilon^n-1}{\varepsilon-1}=0.$$

This gives the result. ❑

We may use complex numbers to select certain sums of coefficients of polynomials. The following problem uses the fact that if k is an integer

$$i^k+i^{k+1}+i^{k+2}+i^{k+3}=i^k(1+i+i^2+i^3)=0 \tag{A.15}$$

492 Example Let

$$(1+x^4+x^8)^{100}=a_0+a_1x+a_2x^2+\cdots+a_{800}x^{800}.$$

Find:

❶ $a_0+a_1+a_2+a_3+\cdots+a_{800}$.

❷ $a_0+a_2+a_4+a_6+\cdots+a_{800}$.

❸ $a_1+a_3+a_5+a_7+\cdots+a_{799}$.

❹ $a_0+a_4+a_8+a_{12}+\cdots+a_{800}$.

❺ $a_1+a_5+a_9+a_{13}+\cdots+a_{797}$.

Solution: ► *Put*

$$p(x)=(1+x^4+x^8)^{100}=a_0+a_1x+a_2x^2+\cdots+a_{800}x^{800}.$$

Then

❶

$$a_0+a_1+a_2+a_3+\cdots+a_{800}=p(1)=3^{100}.$$

❷

$$a_0+a_2+a_4+a_6+\cdots+a_{800}=\frac{p(1)+p(-1)}{2}=3^{100}.$$

❸

$$a_1+a_3+a_5+a_7+\cdots+a_{799}=\frac{p(1)-p(-1)}{2}=0.$$

❹

$$a_0+a_4+a_8+a_{12}+\cdots+a_{800}=\frac{p(1)+p(-1)+p(i)+p(-i)}{4}=2\cdot 3^{100}.$$

❺

$$a_1+a_5+a_9+a_{13}+\cdots+a_{797}=\frac{p(1)-p(-1)-ip(i)+ip(-i)}{4}=0.$$

◄

Homework

A.3.1 Problem Prove that

$$\cos^6 2x=\frac{1}{32}\cos 12x+\frac{3}{16}\cos 8x+\frac{15}{32}\cos 4x+\frac{5}{16}.$$

A.3.2 Problem Prove that

$$\sqrt{3}=\tan\frac{\pi}{9}+4\sin\frac{\pi}{9}.$$

B Binomial Theorem

B.1 Pascal's Triangle

It is well known that

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{B.1}$$

Multiplying this last equality by $a+b$ one obtains

$$(a+b)^3 = (a+b)^2(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

Again, multiplying

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \tag{B.2}$$

by $a+b$ one obtains

$$(a+b)^4 = (a+b)^3(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Dropping the variables, a pattern with the coefficients emerges, a pattern called *Pascal's Triangle.*

Pascal's Triangle

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}$$

Notice that each entry different from 1 is the sum of the two neighbours just above it.
Pascal's Triangle can be used to expand binomials to various powers, as the following examples shew.

493 Example

$$\begin{aligned}
(4x+5)^3 &= (4x)^3 + 3(4x)^2(5) + 3(4x)(5)^2 + 5^3 \\
&= 64x^3 + 240x^2 + 300x + 125
\end{aligned}$$

494 Example

$$\begin{aligned}
(2x-y^2)^4 &= (2x)^4 + 4(2x)^3(-y^2) + 6(2x)^2(-y^2)^2 + \\
&\quad +4(2x)(-y^2)^3 + (-y^2)^4 \\
&= 16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8
\end{aligned}$$

495 Example

$$\begin{aligned}
(2+i)^5 &= 2^5 + 5(2)^4(i) + 10(2)^3(i)^2 + \\
&\quad +10(2)^2(i)^3 + 5(2)(i)^4 + i^5 \\
&= 32 + 80i - 80 - 40i + 10 + i \\
&= -38 + 39i
\end{aligned}$$

496 Example

$$\begin{aligned}(\sqrt{3}+\sqrt{5})^4 &= (\sqrt{3})^4+4(\sqrt{3})^3(\sqrt{5})\\ &\qquad +6(\sqrt{3})^2(\sqrt{5})^2+4(\sqrt{3})(\sqrt{5})^3+(\sqrt{5})^4\\ &= 9+12\sqrt{15}+90+20\sqrt{15}+25\\ &= 124+32\sqrt{15}\end{aligned}$$

497 Example Given that $a-b=2, ab=3$ find a^3-b^3.

Solution: ► *One has*

$$\begin{aligned}8 &= 2^3\\ &= (a-b)^3\\ &= a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab(a-b)\\ &= a^3-b^3-18,\end{aligned}$$

whence $a^3-b^3=26$.
Aliter: *Observe that* $4=2^2=(a-b)^2=a^2+b^2-2ab=a^2-b^2-6$, *whence* $a^2+b^2=10$. *This entails that*

$$a^3-b^3=(a-b)(a^2+ab+b^2)=(2)(10+3)=26,$$

as before. ◄

B.2 Homework

B.2.1 Problem Expand

1. $(x-4y)^3$
2. $(x^3+y^2)^4$
3. $(2+3x)^3$
4. $(2i-3)^4$
5. $(2i+3)^4+(2i-3)^4$
6. $(2i+3)^4-(2i-3)^4$
7. $(\sqrt{3}-\sqrt{2})^3$
8. $(\sqrt{3}+\sqrt{2})^3+(\sqrt{3}-\sqrt{2})^3$
9. $(\sqrt{3}+\sqrt{2})^3-(\sqrt{3}-\sqrt{2})^3$

B.2.2 Problem Prove that

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

Prove that

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$$

Generalise.

B.2.3 Problem Compute $(x+2y+3z)^2$.

B.2.4 Problem Given that $a+2b=-8,\ ab=4$, find (i) a^2+4b^2, (ii) a^3+8b^3, (iii) $\dfrac{1}{a}+\dfrac{1}{2b}$.

B.2.5 Problem The sum of the squares of three consecutive positive integers is 21170. Find the sum of the cubes of those three consecutive positive integers.

B.2.6 Problem What is the coefficient of x^4y^6 in

$$(x\sqrt{2}-y)^{10}?$$

Answer: 840.

B.2.7 Problem Expand and simplify

$$(\sqrt{1-x^2}+1)^7-(\sqrt{1-x^2}-1)^7.$$

C Sequences and Series

C.1 Sequences

498 Definition A sequence of real numbers is a function whose domain is the set of natural numbers and whose output is a subset of the real numbers. We usually denote a sequence by one of the notations

$$a_0, a_1, a_2, \ldots,$$

or

$$\{a_n\}_{n=0}^{+\infty}.$$

Sometimes we may not start at $n = 0$. In that case we may write

$$a_m, a_{m+1}, a_{m+2}, \ldots,$$

or

$$\{a_n\}_{n=m}^{+\infty},$$

where m is a non-negative integer.

We will be mostly interested in two types of sequences: sequences that have an explicit formula for their n-th term and sequences that are defined recursively.

499 Example Let $a_n = 1 - \frac{1}{2^n}, n = 0, 1, \ldots$. Then $\{a_n\}_{n=0}^{+\infty}$ is a sequence for which we have an explicit formula for the n-th term. The first five terms are

$$\begin{array}{lllll} a_0 & = & 1-\frac{1}{2^0} & = & 0, \\ a_1 & = & 1-\frac{1}{2^1} & = & \frac{1}{2}, \\ a_2 & = & 1-\frac{1}{2^2} & = & \frac{3}{4}, \\ a_3 & = & 1-\frac{1}{2^3} & = & \frac{7}{8}, \\ a_4 & = & 1-\frac{1}{2^4} & = & \frac{15}{16}. \end{array}$$

500 Example Let

$$x_0 = 1, \quad x_n = \left(1 + \frac{1}{n}\right) x_{n-1}, \quad n = 1, 2, \ldots.$$

Then $\{x_n\}_{n=0}^{+\infty}$ is a sequence recursively defined. The terms $x_1, x_2, \ldots, x_5$ are

$$\begin{array}{lllll} x_1 & = & \left(1+\frac{1}{1}\right)x_0 & = & 2, \\ x_2 & = & \left(1+\frac{1}{2}\right)x_1 & = & 3, \\ x_3 & = & \left(1+\frac{1}{3}\right)x_2 & = & 4, \\ x_4 & = & \left(1+\frac{1}{4}\right)x_3 & = & 5, \\ x_5 & = & \left(1+\frac{1}{5}\right)x_4 & = & 6. \end{array}$$

You might conjecture that an explicit formula for x_n is $x_n = n + 1$, and you would be right!

501 Definition A sequence $\{a_n\}_{n=0}^{+\infty}$ is said to be *increasing* if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$[1] and *strictly increasing* if $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$[2]

Similarly, a sequence $\{a_n\}_{n=0}^{+\infty}$ is said to be *decreasing* if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$[3] and *strictly decreasing* if $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$[4]

A sequence is *monotonic* if is either increasing, strictly increasing, decreasing, or strictly decreasing.

502 Example Recall that $0! = 1$, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, etc. Prove that the sequence $x_n = n!, n = 0, 1, 2, \ldots$ is strictly increasing for $n \geq 1$.

Solution: ▶ *For $n > 1$ we have*

$$x_n = n! = n(n-1)! = nx_{n-1} > x_{n-1},$$

since $n > 1$. This proves that the sequence is strictly increasing. ◀

503 Example Prove that the sequence $x_n = 2 + \dfrac{1}{2^n}$, $n = 0, 1, 2, \ldots$ is strictly decreasing.

Solution: ▶ *We have*

$$\begin{aligned} x_{n+1} - x_n &= \left(2 + \frac{1}{2^{n+1}}\right) - \left(2 + \frac{1}{2^n}\right) \\ &= \frac{1}{2^{n+1}} - \frac{1}{2^n} \\ &= -\frac{1}{2^{n+1}} \\ &< 0, \end{aligned}$$

whence

$$x_{n+1} - x_n < 0 \implies x_{n+1} < x_n,$$

i.e., the sequence is strictly decreasing. ◀

504 Example Prove that the sequence $x_n = \dfrac{n^2+1}{n}$, $n = 1, 2, \ldots$ is strictly increasing.

Solution: ▶ *First notice that* $\dfrac{n^2+1}{n} = n + \dfrac{1}{n}$. *Now,*

$$\begin{aligned} x_{n+1} - x_n &= \left(n + 1 + \frac{1}{n+1}\right) - \left(n + \frac{1}{n}\right) \\ &= 1 + \frac{1}{n+1} - \frac{1}{n} \\ &= 1 - \frac{1}{n(n+1)} \\ &> 0, \end{aligned}$$

since from 1 we are subtracting a proper fraction less than 1. Hence

$$x_{n+1} - x_n > 0 \implies x_{n+1} > x_n,$$

i.e., the sequence is strictly increasing. ◀

[1] Some people call these sequences *non-decreasing*.
[2] Some people call these sequences *increasing*.
[3] Some people call these sequences *non-increasing*.
[4] Some people call these sequences *decreasing*.

505 Definition A sequence $\{x_n\}_{n=0}^{+\infty}$ is said to be *bounded* if eventually the absolute value of every term is smaller than a certain positive constant. The sequence is *unbounded* if given an arbitrarily large positive real number we can always find a term whose absolute value is greater than this real number.

506 Example Prove that the sequence $x_n = n!, n = 0, 1, 2, \ldots$ is unbounded.

Solution: ▶ *Let $M > 0$ be a large real number. Then its integral part $\lfloor M \rfloor$ satisfies the inequality $M - 1 < \lfloor M \rfloor \leq M$ and so $\lfloor M \rfloor + 1 > M$. We have*

$$x_{\lfloor M \rfloor + 1} = (\lfloor M \rfloor + 1)! = (\lfloor M \rfloor + 1)(\lfloor M \rfloor)(\lfloor M \rfloor - 1) \cdots 2 \cdot 1 > M,$$

since the first factor is greater than M and the remaining factors are positive integers.

◀

507 Example Prove that the sequence $a_n = \dfrac{n+1}{n}$, $n = 1, 2, \ldots,$ is bounded.

Solution: ▶ *Observe that $a_n = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$. Since $\dfrac{1}{n}$ strictly decreases, each term of a_n becomes smaller and smaller. This means that each term is smaller that $a_1 = 1 + \dfrac{1}{2}$. Thus $a_n < 2$ for $n \geq 2$ and the sequence is bounded.* ◀

Homework

C.1.1 Problem Find the first five terms of the following sequences.

1. $x_n = 1 + (-2)^n, n = 0, 1, 2, \ldots$
2. $x_n = 1 + (-\frac{1}{2})^n, n = 0, 1, 2, \ldots$
3. $x_n = n! + 1, n = 0, 1, 2, \ldots$
4. $x_n = \dfrac{1}{n! + (-1)^n}, n = 2, 3, 4, \ldots$
5. $x_n = \left(1 + \dfrac{1}{n}\right)^n, n = 1, 2, \ldots,$

C.1.2 Problem Decide whether the following sequences are eventually monotonic or non-monotonic. Determine whether they are bounded or unbounded.

1. $x_n = n$, $n = 0, 1, 2, \ldots$
2. $x_n = (-1)^n n$, $n = 0, 1, 2, \ldots$
3. $x_n = \dfrac{1}{n!}$, $n = 0, 1, 2, \ldots$
4. $x_n = \dfrac{n}{n+1}$, $n = 0, 1, 2, \ldots$
5. $x_n = n^2 - n$, $n = 0, 1, 2, \ldots$
6. $x_n = (-1)^n$, $n = 0, 1, 2, \ldots$
7. $x_n = 1 - \dfrac{1}{2^n}$, $n = 0, 1, 2, \ldots$
8. $x_n = 1 + \dfrac{1}{2^n}$, $n = 0, 1, 2, \ldots$

C.2 Convergence and Divergence

We are primarily interested in the behaviour that a sequence $\{a_n\}_{n=0}^{+\infty}$ exhibits as n gets larger and larger. First some shorthand.

508 Definition The notation $n \to +\infty$ means that the natural number n increases or tends towards $+\infty$, that is, that it becomes bigger and bigger.

509 Definition We say that the sequence $\{x_n\}_{n=0}^{+\infty}$ *converges*[5] to a limit L, written $x_n \to L$ as $n \to +\infty$, if eventually all terms after a certain term are closer to L by any preassigned distance. A sequence which does not converge is said to *diverge*.

To illustrate the above definition, some examples are in order.

[5]This definition is necessarily imprecise, as we want to keep matters simple. A more precise definition is the following: we say that a sequence $c_n, n = 0, 1, 2, \ldots$ *converges* to L (written $c_n \to L$) as $n \to +\infty$, if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $|c_n - L| < \varepsilon\ \forall n > N$. We say that a sequence $d_n, n = 0, 1, 2, \ldots$ diverges to $+\infty$ (written $d_n \to +\infty$) as $n \to +\infty$, if $\forall M > 0\ \exists N \in \mathbb{N}$ such that $d_n > M\ \forall n > N$. A sequence $f_n, n = 0, 1, 2, \ldots$ diverges to $-\infty$ if the sequence $-f_n, n = 0, 1, 2, \ldots$ converges to $+\infty$.

510 Example The constant sequence

$$1, 1, 1, 1, \ldots$$

converges to 1.

511 Example Consider the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots,$$

We claim that $\frac{1}{n} \to 0$ as $n \to +\infty$. Suppose we wanted terms that get closer to 0 by at least $.00001 = \frac{1}{10^5}$. We only need to look at the 100000-term of the sequence: $\frac{1}{100000} = \frac{1}{10^5}$. Since the terms of the sequence get smaller and smaller, any term after this one will be within .00001 of 0. We had to wait a long time—till after the 100000-th term—but the sequence eventually did get closer than .00001 to 0. The same argument works for any distance, no matter how small, so we can eventually get arbitrarily close to 0.[6].

512 Example The sequence

$$0, 1, 4, 9, 16, \ldots, n^2, \ldots$$

diverges to $+\infty$, as the sequence gets arbitrarily large.[7]

513 Example The sequence

$$1, -1, 1, -1, 1, -1, \ldots, (-1)^n, \ldots$$

has no limit (diverges), as it bounces back and forth from -1 to $+1$ infinitely many times.

514 Example The sequence

$$0, -1, 2, -3, 4, -5, \ldots, (-1)^n n, \ldots,$$

has no limit (diverges), as it is unbounded and alternates back and forth positive and negative values..

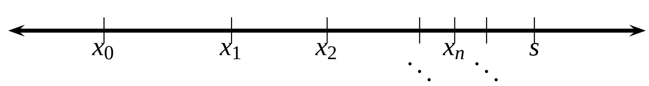


Figure C.1: Theorem 515.

When is it guaranteed that a sequence of real numbers has a limit? We have the following result.

[6]A rigorous proof is as follows. If $\varepsilon > 0$ is no matter how small, we need only to look at the terms after $N = \lfloor \frac{1}{\varepsilon} + 1 \rfloor$ to see that, indeed, if $n > N$, then

$$s_n = \frac{1}{n} < \frac{1}{N} = \frac{1}{\lfloor \frac{1}{\varepsilon} + 1 \rfloor} < \varepsilon.$$

Here we have used the inequality

$$t - 1 < \lfloor t \rfloor \leq t, \quad \forall t \in \mathbb{R}.$$

[7]A rigorous proof is as follows. If $M > 0$ is no matter how large, then the terms after $N = \lfloor \sqrt{M} \rfloor + 1$ satisfy ($n > N$)

$$t_n = n^2 > N^2 = (\lfloor \sqrt{M} \rfloor + 1)^2 > M.$$

515 Theorem Every bounded increasing sequence $\{a_n\}_{n=0}^{+\infty}$ of real numbers converges to its supremum. Similarly, every bounded decreasing sequence of real numbers converges to its infimum.

Proof: *The idea of the proof is sketched in figure C.1. By virtue of Axiom* **??**, *the sequence has a supremum s. Every term of the sequence satisfies $a_n \leq s$. We claim that eventually all the terms of the sequence are closer to s than a preassigned small distance $\varepsilon > 0$. Since $s-\varepsilon$ is not an upper bound for the sequence, there must be a term of the sequence, say a_{n_0} with $s-\varepsilon \leq a_{n_0}$ by virtue of the Approximation Property Theorem* **??**. *Since the sequence is increasing, we then have*

$$s-\varepsilon \leq a_{n_0} \leq a_{n_0+1} \leq a_{n_0+2} \leq a_{n_0+2} \leq \ldots \leq s,$$

which means that after the n_0-th term, we get to within ε of s.

To obtain the second half of the theorem, we simply apply the first half to the sequence $\{-a_n\}_{n=0}^{+\infty}$. ❑

Homework

C.2.1 Problem Give plausible arguments to convince yourself that

1. $\frac{1}{2^n} \to 0$ as $n \to +\infty$
2. $2^n \to +\infty$ as $n \to +\infty$
3. $\frac{1}{n!} \to 0$ as $n \to +\infty$
4. $\frac{n+1}{n} \to 1$ as $n \to +\infty$
5. $(\frac{2}{3})^n \to 0$ as $n \to +\infty$
6. $(\frac{3}{2})^n \to +\infty$ as $n \to +\infty$
7. the sequence $(-2)^n, n = 0, 1, \ldots$ diverges as $n \to +\infty$
8. $\frac{n}{2^n} \to 0$ as $n \to +\infty$
9. $\frac{2^n}{n} \to +\infty$ as $n \to +\infty$
10. the sequence $1+(-1)^n, n = 0, 1, \ldots$ diverges as $n \to +\infty$

C.3 Finite Geometric Series

516 Definition A *geometric sequence* or *progression* is a sequence of the form

$$a,\ ar, ar^2,\ ar^3,\ ar^4, \ldots,$$

that is, every term is produced from the preceding one by multiplying a fixed number. The number r is called the *common ratio*.

1. *Trivially, if $a = 0$, then every term of the progression is 0, a rather uninteresting case.*
2. *If $ar \neq 0$, then the common ratio can be found by dividing any term by that which immediately precedes it.*
3. *The n-th term of the progression*
$$a,\ ar, ar^2,\ ar^3,\ ar^4, \ldots,$$
is ar^{n-1}.

517 Example Find the 35-th term of the geometric progression

$$\frac{1}{\sqrt{2}},\ -2,\ \frac{8}{\sqrt{2}}, \ldots.$$

Solution: ▶ *The common ratio is $-2 \div \frac{1}{\sqrt{2}} = -2\sqrt{2}$. Hence the 35-th term is $\frac{1}{\sqrt{2}}(-2\sqrt{2})^{34} = \frac{2^{51}}{\sqrt{2}} = 1125899906842624\sqrt{2}$.* ◀

518 Example The fourth term of a geometric progression is 24 and its seventh term is 192. Find its second term.

Solution: ▶ *We are given that* $ar^3 = 24$ *and* $ar^6 = 192$, *for some* a *and* r. *Clearly,* $ar \neq 0$, *and so we find*

$$\frac{ar^6}{ar^3} = r^3 = \frac{192}{24} = 8,$$

whence $r = 2$. *Now,* $a(2)^3 = 24$, *giving* $a = 3$. *The second term is thus* $ar = 6$. ◀

519 Example Find the sum

$$2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{64}.$$

Estimate (without a calculator!) how big this sum is.

Solution: ▶ *Let*

$$S = 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{64}.$$

Observe that the common ratio is 2. *We multiply* S *by* 2 *and notice that every term, with the exception of the last, appearing on this new sum also appears on the first sum. We subtract* S *from* $2S$*:*

$$\begin{array}{lclcccccccccccc}
S & = & 2 & + & 2^2 & + & 2^3 & + & 2^4 & + & \cdots & + & 2^{64} & & \\
2S & = & & & 2^2 & + & 2^3 & + & 2^4 & + & \cdots & + & 2^{64} & + & 2^{65} \\
\hline
2S - S & = & -2 + 2^{65} & & & & & & & & & & & &
\end{array}$$

Thus $S = 2^{65} - 2$. *To estimate this sum observe that* $2^{10} = 1024 \approx 10^3$. *Therefore*

$$2^{65} = (2^{10})^6 \cdot (2^5) = 32(2^{10})^6 \approx 32(10^3)^6 = 32 \times 10^{18} = 3.2 \times 10^{19}.$$

The exact answer (obtained via Maple ®), is

$$2^{65} - 2 = 36893488147419103230.$$

My pocket calculator yields $3.689348815 \times 10^{19}$. *Our estimate gives the right order of decimal places.* ◀

1. *If a chess player is paid* \$2 *for the first square of a chess board,* \$4 *for the second square,* \$8 *for the third square, etc., after reaching the* 64*-th square he would be paid* \$36893488147419103230. *Query: After which square is his total more than* \$1000000*?*
2. *From the above example, the sum of a geometric progression with positive terms and common ratio* $r > 1$ *grows rather fast rather quickly.*

520 Example Sum

$$\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^{99}}.$$

Solution: ▶ *Put*

$$S = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^{99}}.$$

Then

$$\frac{1}{3}S = \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \cdots + \frac{2}{3^{100}}.$$

Subtracting,

$$S - \frac{1}{3}S = \frac{2}{3}S = \frac{2}{3} - \frac{2}{3^{100}}.$$

It follows that

$$S = \frac{3}{2}\left(\frac{2}{3} - \frac{2}{3^{100}}\right) = 1 - \frac{1}{3^{99}}.$$

◀

The sum of the first two terms of the series in example 520 is $\frac{2}{3}+\frac{2}{3^2}=\frac{8}{9}$, which, though close to 1 is not as close as the sum of the first 99 terms. A geometric progression with positive terms and common ratio $0<r<1$ has a sum that grows rather slowly.

To close this section we remark that the approximation $2^{10}\approx 1000$ is a useful one. It is nowadays used in computer lingo, where a kilobyte is 1024 bytes—"kilo" is a Greek prefix meaning "thousand."

521 Example Without using a calculator, determine which number is larger: 2^{900} or 3^{500}.

Solution: ► *The idea is to find a power of 2 close to a power of 3. One readily sees that $2^3=8<9=3^2$. Now, raising both sides to the 250-th power,*

$$2^{750}=(2^3)^{250}<(3^2)^{250}=3^{500}.$$

The inequality just obtained is completely useless, it does not answer the question addressed in the problem. However, we may go around this with a similar idea. Observe that $9<8\sqrt{2}$: for, if $9\geq 8\sqrt{2}$, squaring both sides we would obtain $81>128$, a contradiction. Raising $9<8\sqrt{2}$ to the 250-th power we obtain

$$3^{500}=(3^2)^{250}<(8\sqrt{2})^{250}=2^{875}<2^{900},$$

whence 2^{900} is greater. ◄

You couldn't solve example 521 using most pockets calculators and the mathematical tools you have at your disposal (unless you were really *clever!). Later in this chapter we will see how to solve this problem using logarithms.*

Homework

C.3.1 Problem Find the 17-th term of the geometric sequence

$$-\frac{2}{3^{17}},\frac{2}{3^{16}},-\frac{2}{3^{15}},\cdots.$$

C.3.2 Problem The 6-th term of a geometric progression is 20 and the 10-th is 320. Find the absolute value of its third term.

C.3.3 Problem Find the sum of the following geometric series.

1. $$1+3+3^2+3^3+\cdots+3^{49}.$$
2. If $y\neq 1$, $$1+y+y^2+y^3+\cdots+y^{100}.$$
3. If $y\neq 1$, $$1-y+y^2-y^3+y^4-y^5+\cdots-y^{99}+y^{100}.$$
4. If $y\neq 1$, $$1+y^2+y^4+y^6+\cdots+y^{100}.$$

C.3.4 Problem A colony of amoebas[8] is put in a glass at 2 : 00 PM. One second later each amoeba divides in two. The next second, the present generation divides in two again, etc.. After one minute, the glass is full. When was the glass half-full?

C.3.5 Problem Without using a calculator: which number is greater 2^{30} or 30^2?

C.3.6 Problem In this problem you may use a calculator. Legend says that the inventor of the game of chess asked the Emperor of China to place a grain of wheat on the first square of the chessboard, 2 on the second square, 4 on the third square, 8 on the fourth square, etc.. (1) How many grains of wheat are to be put on the last (64-th) square?, (2) How many grains, total, are needed in order to satisfy the greedy inventor?, (3) Given that 15 grains of wheat weigh approximately one gramme, what is the approximate weight, in kg, of wheat needed?, (4) Given that the annual production of wheat is 350 million tonnes, how many years, approximately, are needed in order to satisfy the inventor (assume that production of wheat stays constant)[9].

C.3.7 Problem Prove that

$$1+2\cdot 5+3\cdot 5^2+4\cdot 5^3+\cdots+99\cdot 5^{100}=\frac{99\cdot 5^{101}}{4}-\frac{5^{101}-1}{16}.$$

C.3.8 Problem Shew that

$$1+x+x^2+\cdots+x^{1023}=(1+x)(1+x^2)(1+x^4)\cdots(1+x^{256})(1+x^{512}).$$

C.3.9 Problem Prove that

$$1+x+x^2+\cdots+x^{80}=(x^{54}+x^{27}+1)(x^{18}+x^9+1)(x^6+x^3+1)(x^2+x+1).$$

[8] Why are amoebas bad mathematicians? Because they divide to multiply!

[9] Depending on your ethnic preference, the ruler in this problem might be an Indian maharajah or a Persian shah, but never an American businessman!!!

C.4 Infinite Geometric Series

522 Definition Let

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

be the sequence of partial sums of a geometric progression. We say that the *infinite* geometric sum

$$a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + \cdots$$

converges to a finite number s if $|s_n - s| \to 0$ as $n \to +\infty$. We say that infinite sum

$$a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + \cdots$$

diverges if there is no finite number to which the sequence of partial sums converges.

523 Lemma If $0 < a < 1$ then $a^n \to 0$ as $n \to 0$.

Proof: *Observe that by multiplying through by a we obtain*

$$0 < a < 1 \implies 0 < a^2 < a \implies 0 < a^3 < a^2 \implies \ldots$$

and so

$$0 < \ldots < a^n < a^{n-1} < \ldots < a^3 < a^2 < a < 1,$$

that is, the sequence is decreasing and bounded. By Theorem 515 the sequence converges to its infimum $\inf_{n \geq 0} a^n = 0$. ❑

524 Theorem Let $a,\ ar,\ ar^2,\ \ldots$ with $|r| \neq 1$, be a geometric progression. Then

1. The sum of its first n terms is
$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1-r},$$
2. If $|r| < 1$, the infinite sum converges to
$$a + ar + ar^2 + \cdots = \frac{a}{1-r},$$
3. If $|r| > 1$, the infinite sum diverges.

Proof: *Put*

$$S = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Then

$$rS = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Subtracting,

$$S - rS = S(1-r) = a - ar^n.$$

Since $|r| \neq 1$ *we may divide both sides of the preceding equality in order to obtain*

$$S = \frac{a - ar^n}{1-r},$$

proving the first statement of the theorem.

Now, if $|r| < 1$, *then* $|r|^n \to 0$ *as* $n \to +\infty$ *by virtue of Lemma 523, and if* $|r| > 1$, *then* $|r|^n \to +\infty$ *as* $n \to +\infty$. *The second and third statements of the theorem follow from this.* ❑

We have thus created a dichotomy amongst infinite geometric sums. If their common ratio is smaller than 1 *in absolute value, the infinite geometric sum converges. Otherwise, the sum diverges.*

525 Example Find the sum of the infinite geometric series

$$\frac{3}{5^3}-\frac{3}{5^4}+\frac{3}{5^5}-\frac{3}{5^6}+\cdots.$$

Solution: ▶ *We have* $a=\frac{3}{5^3}, r=-\frac{1}{5}$ *in Theorem 524. Therefore*

$$\frac{3}{5^3}-\frac{3}{5^4}+\frac{3}{5^5}-\frac{3}{5^6}+\cdots=\frac{\frac{3}{5^3}}{1-\left(-\frac{1}{5}\right)}=\frac{1}{50}.$$

◀

526 Example Find the rational number which is equivalent to the repeating decimal $0.23\overline{45}$.

Solution: ▶

$$0.23\overline{45}=\frac{23}{10^2}+\frac{45}{10^4}+\frac{45}{10^6}+\cdots=\frac{23}{10^2}+\frac{\frac{45}{10^4}}{1-\frac{1}{10^2}}=\frac{23}{100}+\frac{1}{220}=\frac{129}{550}.$$

◀

The geometric series above did not start till the second term of the sum.

527 Example A celestial camel is originally at the point $(0,0)$ on the Cartesian Plane. The camel is told by a Djinn that if it wanders 1 unit right, $1/2$ unit up, $1/4$ unit left, $1/8$ unit down, $1/16$ unit right, and so, ad infinitum, then it will find a houris. What are the coordinate points of the houris?

Solution: ▶ *Let the coordinates of the houris be* (X,Y). *Then*

$$X=\frac{1}{4}+\frac{1}{4^2}-\frac{1}{4^3}+\cdots=\frac{1}{1-\left(-\frac{1}{4}\right)}=\frac{4}{5},$$

and

$$Y=\frac{1}{2}-\frac{1}{2^3}+\frac{1}{2^5}-\frac{1}{2^7}\cdots=\frac{\frac{1}{2}}{1-\left(-\frac{1}{4}\right)}=\frac{2}{5}.$$

◀

528 Example What is wrong with the statement

$$1+2+2^2+2^3+\cdots=\frac{1}{1-2}=-1?$$

Notice that the sinistral side is positive and the dextral side is negative.

Solution: ▶ *The geometric sum diverges, as the common ratio 2 is* >1, *so we may not apply the formula for an infinite geometric sum. There is an interpretation (called* convergence in the sense of Abel*), where statements like the one above do make sense.* ◀

Homework

C.4.1 Problem Find the sum of the given infinite geometric series.

1.
$$\frac{8}{5}+1+\frac{5}{8}+\cdots$$

2.
$$0.9+0.03+0.001+\cdots$$

3.
$$\frac{3+2\sqrt{2}}{3-2\sqrt{2}}+1+\frac{3-2\sqrt{2}}{3+2\sqrt{2}}+\cdots$$

4.
$$\frac{\sqrt{3}}{\sqrt{2}}+\frac{\sqrt{2}}{3}+\frac{2\sqrt{2}}{9\sqrt{3}}+\cdots$$

5.
$$1 + \frac{\sqrt{5}-1}{2} + \left(\frac{\sqrt{5}-1}{2}\right)^2 + \cdots$$

6.
$$1 + 10 + 10^2 + 10^3 + \cdots$$

7.
$$1 - x + x^2 - x^3 + \cdots, \quad |x| < 1.$$

8.
$$\frac{\sqrt{3}}{\sqrt{3}+1} + \frac{\sqrt{3}}{\sqrt{3}+3} + \cdots$$

9.
$$x - y + \frac{y^2}{x} - \frac{y^3}{x^2} + \frac{y^4}{x^3} - \frac{y^5}{x^4} + \cdots,$$

with $|y| < |x|$.

C.4.2 Problem Give rational numbers (that is, the quotient of two integers), equivalent to the repeating decimals below.

1. $0.\overline{3}$
2. $0.\overline{6}$
3. $0.2\overline{5}$
4. $2.12\overline{35}$
5. $0.\overline{428571}$

C.4.3 Problem Give an example of an infinite series with all positive terms, adding to 666.

D Old Exam Questions

D.1 Multiple-Choice

D.1.1 Real Numbers

1. The infinite repeating decimal $0.102102\ldots = 0.\overline{102}$ as a quotient of two integers is
   (A) $\dfrac{15019}{147098}$ (B) $\dfrac{34}{333}$ (C) $\dfrac{51}{500}$ (D) $\dfrac{101}{999}$ (E) none of these

2. Express the infinite repeating decimal $0.424242\ldots = 0.\overline{42}$ as a fraction.
   (A) $\dfrac{21}{50}$ (B) $\dfrac{14}{33}$ (C) $\dfrac{7}{15}$ (D) $\dfrac{14}{333}$ (E) none of these

3. Write the infinite repeating decimal as a fraction: $0.121212\ldots = 0.\overline{12}$.
   (A) $\dfrac{4}{33}$ (B) $\dfrac{3}{25}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{102}{333}$ (E) none of these

4. Let $a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$. How many of the following are *necessarily* irrational numbers?
   $$I : a+b, \qquad II : ab, \qquad III : 1+a+b, \qquad IV : 1+a^2+b^2$$
   (A) exactly one (B) exactly two (C) exactly three (D) all four (E) none

5. Let $a \in \mathbb{Z}$. How many of the following are *necessarily* true?
   $$I : \sqrt{|a|} \in \mathbb{R} \setminus \mathbb{Q}, \qquad II : \sqrt{a^2} \in \mathbb{Z}, \qquad III : \frac{a}{1+|a|} \in \mathbb{Q}, \qquad IV : \sqrt{1+a^2} \in \mathbb{R} \setminus \mathbb{Q}$$
   (A) exactly one (B) exactly two (C) exactly three (D) all four (E) none

D.1.2 Sets on the Line

6. $]-3;2[\ \cap\ [1;3] =$
   (A) $]-3;1[$ (B) $]-3;1]$ (C) $[1;2[$ (D) $]-3;3]$ (E) none of these

7. Determine the set of all real numbers x satisfying the inequality $\dfrac{x+2}{x-1} < 1$.
   (A) $]1;+\infty[$ (B) $]-2;1[$ (C) $]-\infty;1[$ (D) $]-\infty;1]$ (E) none of these

8. $]-3;8] \quad \cap \quad [-8;-3[=$.
   (A) $\{-3\}$ (B) $\varnothing$ (C) $]-8;8]$ (D) $]-8;8[$ (E) none of these

9. Write as a single interval: $]-2;4] \cup [1;5[$.
   (A) $]-2;1[$ (B) $]1;4[$ (C) $]-2;5[$ (D) $[1;4]$ (E) none of these

10. Write as a single interval the following interval difference: $]-5;2[\quad \setminus \quad [-3;3]$.
    (A) $]-5;-3[$ (B) $[-5;-3[$ (C) $[-5;-3]$ (D) $]-5;-3]$ (E) none of these

11. If $\dfrac{x+1}{x(x-1)} \geq 0$ then $x \in$
   (A) $]-\infty;0] \cup [1;+\infty[$
   (B) $[-1;0[\cup]1;+\infty[$
   (C) $[-1;1[\cup]1;+\infty[$
   (D) $]-\infty;0[\cup]0;1[$
   (E) none of these

12. If $\dfrac{3}{x-1} - \dfrac{1}{x} \leq \dfrac{1}{x}$ then $x \in$
   (A) $]-\infty;-2] \cup]0;1[$ (B) $]-2;1[$ (C) $[-2;0[\cup]1;+\infty[$ (D) $]-\infty;+\infty[$ (E) none of these

D.1.3 Absolute Values

Situation: Consider the absolute value expression $|x+2|+|x|-x$. Problems 13 through 17 refer to it.

13. Write $|x+2|+|x|-x$ without absolute values in the interval $]-\infty;-2]$.
   (A) $-x-2$ (B) $x+2$ (C) $-3x-2$ (D) $2-x$ (E) none of these

14. Write $|x+2|+|x|-x$ without absolute values in the interval $[-2;0]$.
   (A) $-x-2$ (B) $x+2$ (C) $-3x-2$ (D) $2-x$ (E) none of these

15. Write $|x+2|+|x|-x$ without absolute values in the interval $[0;+\infty[$.
   (A) $-x-2$ (B) $x+2$ (C) $-3x-2$ (D) $2-x$ (E) none of these

16. If $|x+2|+|x|-x = 2$, then $x \in$
   (A) $\varnothing$ (B) $\{-2\}$ (C) $[-2;0]$ (D) $\{0\}$ (E) none of these

17. If $|x+2|+|x|-x = 3$, then $x \in$
   (A) $\{0,1\}$ (B) $\{-1,0\}$ (C) $[-1;1]$ (D) $\{-1,1\}$ (E) none of these

18. $||\sqrt{2}-2|-2| =$
   (A) $\sqrt{2}$ (B) $\sqrt{2}-4$ (C) $4-\sqrt{2}$ (D) $1+\sqrt{2}$ (E) none of these

19. If $|x+1| = 4$ then
   (A) $x \in \{-5,3\}$ (B) $x \in \{-4,4\}$ (C) $x \in \{-3,5\}$ (D) $x \in \{-5,5\}$ (E) none of these

20. If $-1 < x < 1$ then $|x+1|-|x-1| =$
   (A) 2 (B) -2 (C) $2x$ (D) $-2x$ (E) none of these

21. The set $\{x \in \mathbb{R} : |x+1| < 4\}$ is which of the following intervals?
   (A) $]-4;4[$ (B) $]-5;3[$ (C) $]-3;5[$ (D) $]-1;4[$ (E) none of these

22. If $|x^2-2x| = 1$ then
   (A) $x \in \{1-\sqrt{2}, 1+\sqrt{2}, 2\}$
   (B) $x \in \{1-\sqrt{2}, 1+\sqrt{2}, -1\}$
   (C) $x \in \{-\sqrt{2}, \sqrt{2}\}$
   (D) $x \in \{1-\sqrt{2}, 1+\sqrt{2}, 1\}$
   (E) none of these

Situation: Consider the absolute value expression $|x|+|x-2|$. Problems 23 through 24 refer to it.

23. Which of the following assertions is true?

(A) $|x|+|x-2| = \begin{cases} 2x-2 & \text{if } x \in]-\infty;0] \\ 2 & \text{if } x \in [0;2] \\ -2x+2 & \text{if } x \in [2;+\infty[\end{cases}$

(B) $|x|+|x-2| = \begin{cases} -2x+2 & \text{if } x \in]-\infty;0] \\ 2 & \text{if } x \in [0;2] \\ 2x-2 & \text{if } x \in [2;+\infty[\end{cases}$

(C) $|x|+|x-2| = \begin{cases} -2x+2 & \text{if } x \in]-\infty;-2] \\ 2 & \text{if } x \in [-2;0] \\ 2x-2 & \text{if } x \in [0;+\infty[\end{cases}$

(D) $|x|+|x-2| = \begin{cases} -2x+2 & \text{if } x \in]-\infty;0] \\ -2 & \text{if } x \in [0;2] \\ 2x-2 & \text{if } x \in [2;+\infty[\end{cases}$

(E) none of these

24. If $|x|+|x-2|=3$, then $x \in$

(A) $\varnothing$ (B) $[0;2]$ (C) $\left\{\frac{1}{2}, -\frac{5}{2}\right\}$ (D) $\left\{-\frac{1}{2}, \frac{5}{2}\right\}$ (E) none of these

D.1.4 Sets on the Plane.

25. Find the distance between $(1,-1)$ and $(-1,1)$.

(A) 0 (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$ (E) none of these

26. Find the distance between $(a,-a)$ and $(1,1)$.

(A) $\sqrt{2(1-a)^2}$ (B) $\sqrt{(1-a)^2+(1+a)^2}$ (C) $2\sqrt{(1-a)^2}$ (D) $a\sqrt{2}+2$ (E) none of these

27. What is the distance between the points (a,b) and $(-a,-b)$?

(A) 0 (B) $\sqrt{a^2+b^2}$ (C) $\sqrt{2a^2+2b^2}$ (D) $2\sqrt{a^2+b^2}$ (E) none of these

28. Which one of the following graphs best represents the set

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, \quad x^2 \geq 1\} \ ?$$

Notice that there are four graphs, but five choices.

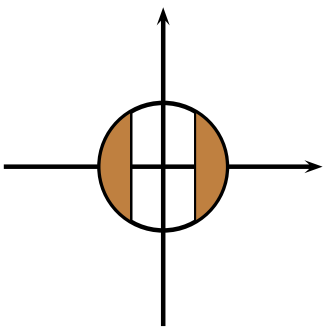


Figure D.1: A

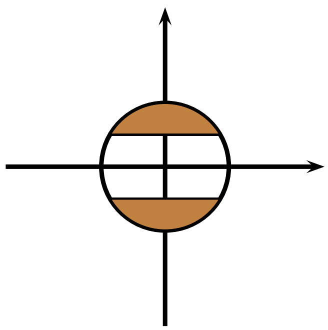


Figure D.2: B

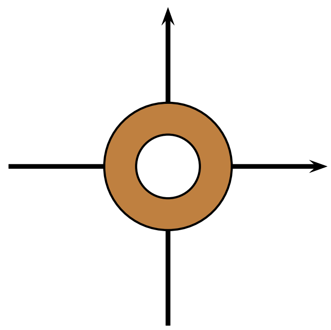


Figure D.3: C

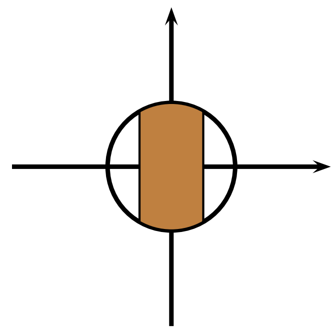


Figure D.4: D

(A) A (B) B (C) C (D) D (E) none of these

29. Which one of the following graphs best represents the set

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1, \quad (x-1)^2 + y^2 \leq 1\} \ ?$$

Notice that there are four graphs, but five choices.

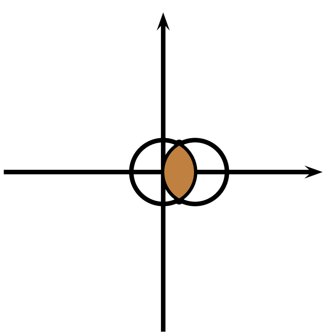


Figure D.5: A

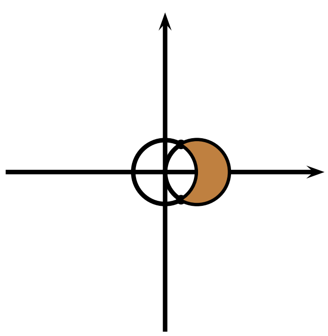


Figure D.6: B

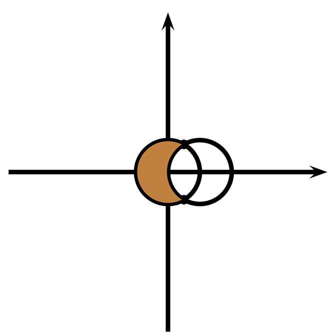


Figure D.7: C

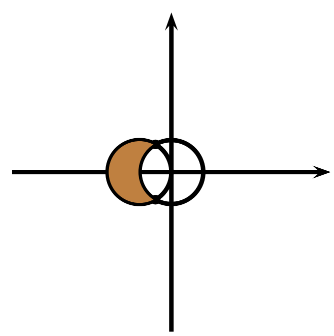


Figure D.8: D

(A) A (B) B (C) C (D) D (E) none of these

30. Which one of the following graphs best represents the set

$$\{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 16, \quad y \ge -x\}\ ?$$

Notice that there are four graphs, but five choices.

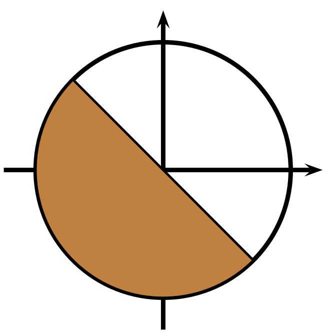


Figure D.9: A

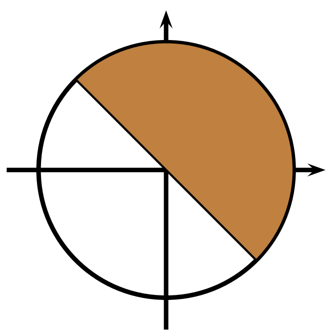


Figure D.10: B

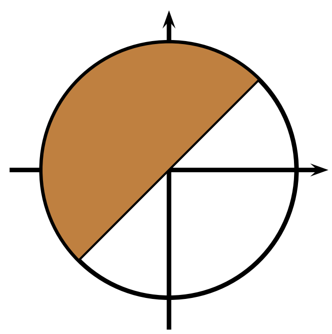


Figure D.11: C

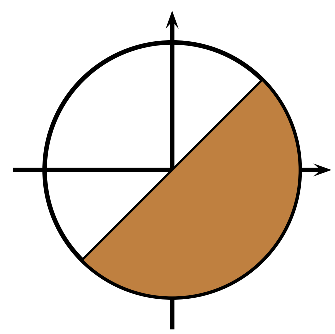


Figure D.12: D

(A) A (B) B (C) C (D) D (E) none of these

31. Which of the following graphs represents the set

$$\{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 4, \quad |x| \ge 1\}?$$

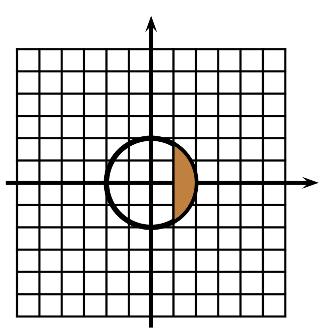


Figure D.13: A

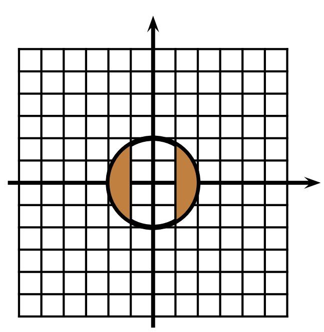


Figure D.14: B

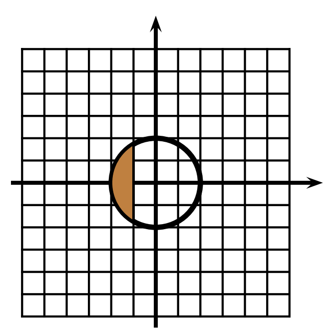


Figure D.15: C

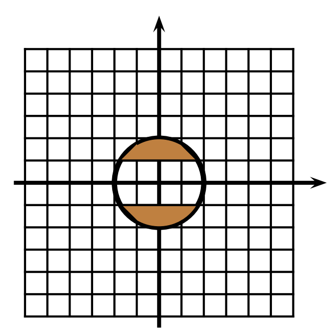


Figure D.16: D

(A) A (B) B (C) C (D) D (E) none of these

32. Which of the following graphs represents the set

$$\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2, \quad 3 \le y \le 4\}?$$

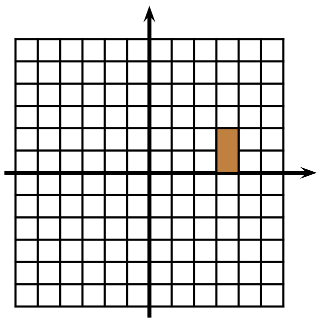


Figure D.17: A

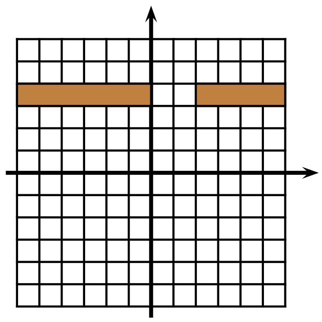


Figure D.18: B

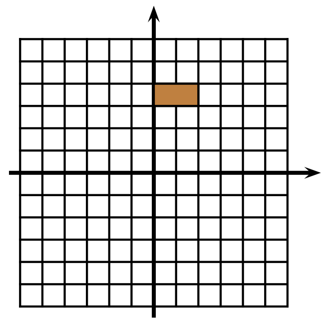


Figure D.19: C

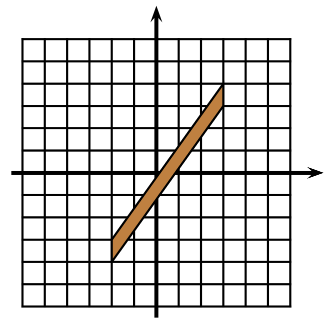


Figure D.20: D

(A) A (B) B (C) C (D) D (E) none of these

D.1.5 Lines

33. The lines with equations $ax + by = c$ and $dx + ey = f$ are perpendicular, where a, b, c, d, e, f are non-zero constants. Which of the following must be true?
(A) $ad - be = 0$ (B) $ad + be = -1$ (C) $ae + bd = -1$ (D) $ae + bd = 0$ (E) $ad + be = 0$

34. If a, b are non-zero real constants, find the equation of the line passing through (a,b) and parallel to the line $L : \frac{x}{a} - \frac{y}{b} = 1$.
(A) $y = \frac{b}{a}x - a$ (B) $y = -\frac{a}{b}x - b$ (C) $y = \frac{a}{b}x + a$ (D) $y = \frac{b}{a}x$ (E) none of these

35. If a, b are non-zero real constants, find the equation of the line passing through (a,b) and perpendicular to the line $L : \frac{x}{a} - \frac{y}{b} = 1$.
(A) $y = -\frac{a}{b}x + b + \frac{a^2}{b}$ (B) $y = -\frac{a}{b}x - b$ (C) $y = \frac{a}{b}x + a$ (D) $y = \frac{b}{a}x + a$ (E) none of these

36. If the points $(1,1)$, $(2,3)$, and $(4,a)$ are on the same line, find the value of a.
(A) 7 (B) -7 (C) 6 (D) 2 (E) none of these

37. If the lines $L : \quad ax - 2y = c$ and $L' : \quad by - x = a$ are parallel, then
(A) $\frac{a}{2} = \frac{1}{b}$ (B) $\frac{a}{2} = -\frac{1}{b}$ (C) $\frac{a}{2} = b$ (D) $\frac{a}{2} = -b$ (E) none of these

38. If the lines $L : \quad ax - 2y = c$ and $L' : \quad by - x = a$ are perpendicular, then
(A) $\frac{a}{2} = \frac{1}{b}$ (B) $\frac{a}{2} = -\frac{1}{b}$ (C) $\frac{a}{2} = b$ (D) $\frac{a}{2} = -b$ (E) none of these

39. Find the equation of the line parallel to $y = mx + k$ and passing through $(1,1)$.
(A) $y = mx + 1$ (B) $y = mx + 1 - m$ (C) $y = mx + m - 1$ (D) $y = mx$ (E) none of these

40. Find the equation of the line perpendicular to $y = mx + k$ and passing through $(1, 1)$.
   (A) $y = -\dfrac{x}{m} - 1 + \dfrac{1}{m}$
   (B) $y = -\dfrac{x}{m} + 1 + \dfrac{1}{m}$
   (C) $y = -\dfrac{x}{m} + 1 - \dfrac{1}{m}$
   (D) $y = -\dfrac{x}{m} - 1 - \dfrac{1}{m}$
   (E) none of these

Problems 41 through 44 refer to the two points $(a, -a)$ and $(1, 1)$.

41. Find the slope of the line joining $(a, -a)$ and $(1, 1)$.
   (A) $\dfrac{1-a}{1+a}$ (B) $\dfrac{1+a}{1-a}$ (C) $\dfrac{1+a}{a-1}$ (D) -1 (E) none of these

42. Find the equation of the line passing through $(a, -a)$ and $(1, 1)$.
   (A) $y = \left(\dfrac{1+a}{1-a}\right)x + \dfrac{2a}{1-a}$
   (B) $y = \left(\dfrac{1+a}{1-a}\right)x$
   (C) $y = \left(\dfrac{1+a}{1-a}\right)x + \dfrac{2a}{a-1}$
   (D) $y = \left(\dfrac{a-1}{a+1}\right)x$
   (E) none of these

43. Find the equation of the line passing through $(0, 0)$ and parallel to the line passing through $(a, -a)$ and $(1, 1)$.
   (A) $y = \left(\dfrac{1+a}{1-a}\right)x + \dfrac{2a}{1-a}$
   (B) $y = \left(\dfrac{1+a}{1-a}\right)x$
   (C) $y = \left(\dfrac{1+a}{1-a}\right)x + \dfrac{2a}{a-1}$
   (D) $y = \left(\dfrac{a-1}{a+1}\right)x$
   (E) none of these

44. Find the equation of the line passing through $(0, 0)$ and perpendicular to the line passing through $(a, -a)$ and $(1, 1)$.
   (A) $y = \left(\dfrac{1-a}{1+a}\right)x$
   (B) $y = \left(\dfrac{1+a}{1-a}\right)x$
   (C) $y = \left(\dfrac{1+a}{1-a}\right)x + \dfrac{2a}{a-1}$
   (D) $y = \left(\dfrac{a-1}{a+1}\right)x$
   (E) none of these

Problems 45 through 48 refer to the following. For a given real parameter u, consider the family of lines L_u given by

$$L_u: \quad (u+1)y + (u-2)x = u.$$

45. For which value of u is L_u horizontal?
   (A) $u = -1$ (B) $u = 2$ (C) $u = \dfrac{1}{3}$ (D) $u = \dfrac{2}{3}$ (E) none of these

46. For which value of u is L_u vertical?

(A) $u=-1$ (B) $u=2$ (C) $u=\frac{1}{3}$ (D) $u=\frac{2}{3}$ (E) none of these

47. For which value of u is L_u parallel to the line $y=2x-1$?

(A) $u=0$ (B) $u=2$ (C) $u=5$ (D) $u=\frac{2}{3}$ (E) none of these

48. For which value of u is L_u perpendicular to the line $y=2x-1$?

(A) $u=-5$ (B) $u=0$ (C) $u=-\frac{1}{2}$ (D) $u=5$ (E) none of these

For a real number parameter u consider the line L_u given by the equation

$$L_u: \quad (u-2)y=(u+1)x+u.$$

Questions 49 to 54 refer to L_u.

49. For which value of u does L_u pass through the point $(-1,1)$?

(A) 1 (B) -1 (C) 2 (D) 3 (E) none of these

50. For which value of u is L_u parallel to the x-axis?

(A) -2 B) 2 (C) -1 (D) 1 (E) none of these

51. For which value of u is L_u parallel to the y-axis?

(A) -2 (B) 2 (C) -1 (D) 1 (E) none of these

52. For which value of u is L_u parallel to the line $2x-y=2$?

(A) 5 (B) 0 (C) -3 (D) $\frac{1}{3}$ (E) none of these

53. For which value of u is L_u perpendicular to the line $2x-y=2$?

(A) 5 (B) 0 (C) $\frac{1}{3}$ (D) $-\frac{1}{3}$ (E) none of these

54. Which of the following points is on every line L_u regardless the value of u?

(A) $(-1,2)$ (B) $(2,-1)$ (C) $(\frac{1}{3},-\frac{2}{3})$ (D) $(-\frac{2}{3},\frac{1}{3})$ (E) none of these

D.1.6 Absolute Value Curves

Situation: *Problems 55 and 56 refer to the curve* $y = |x-2| + |x+1|$.

55. Write $y = |x-2| + |x+1|$ without absolute values.

(A) $y = \begin{cases} -2x+1 & \text{if } x \le -1 \\ 3 & \text{if } -1 \le x \le 2 \\ 2x-1 & \text{if } x \ge 2 \end{cases}$

(B) $y = \begin{cases} -2x+3 & \text{if } x \le -1 \\ 1 & \text{if } -1 \le x \le 2 \\ 2x-3 & \text{if } x \ge 2 \end{cases}$

(C) $y = \begin{cases} -2x-3 & \text{if } x \le -1 \\ 3 & \text{if } -1 \le x \le 2 \\ 2x+3 & \text{if } x \ge 2 \end{cases}$

(D) $y = \begin{cases} -2x-3 & \text{if } x \le -1 \\ 1 & \text{if } -1 \le x \le 2 \\ 2x+3 & \text{if } x \ge 2 \end{cases}$

(E) none of these

56. Which graph most resembles the curve $y = |x-2| + |x+1|$?

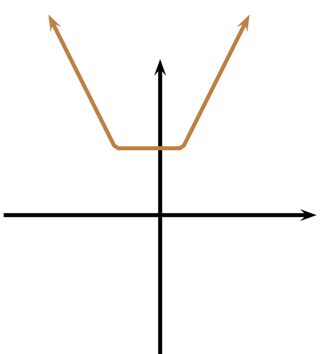


Figure D.21: A

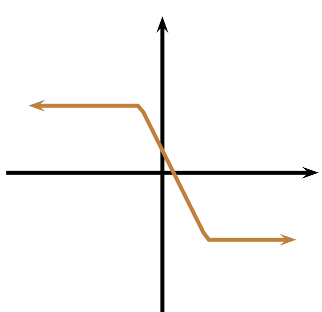


Figure D.22: B

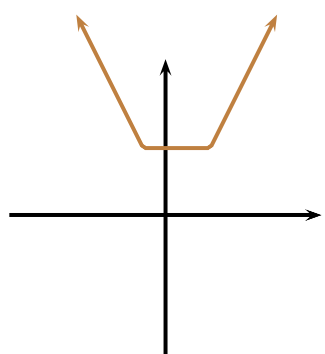


Figure D.23: C

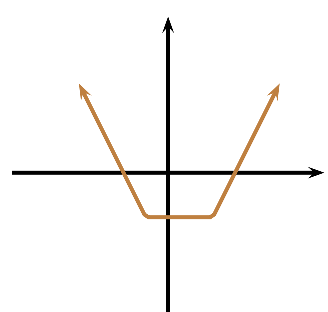


Figure D.24: D

(A) (B) (C) (D) (E) none of these

57. Which graph most resembles the curve $y = |x-2| - |x+1|$?

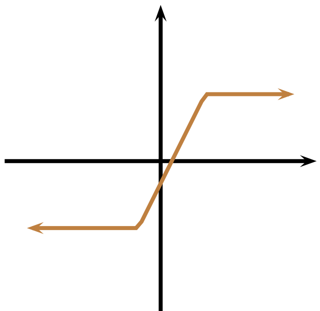


Figure D.25: A

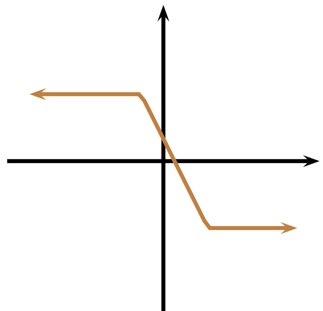


Figure D.26: B

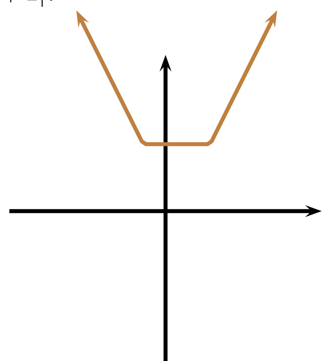


Figure D.27: C

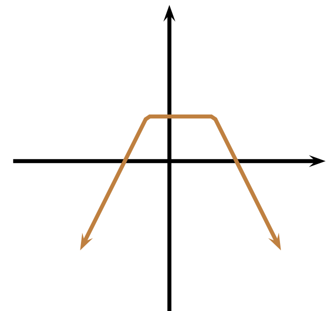


Figure D.28: D

(A) (B) (C) (D) (E) none of these

D.1.7 Circles and Semicircles

58. The point $A(1,2)$ lies on the circle $\mathscr{C} : (x+1)^2 + (y-1)^2 = 5$. Which of the following points is diametrically opposite to A on $\mathscr{C}$?

(A) $(-1,-2)$ (B) $(-3,0)$ (C) $(0,3)$ (D) $(0,\sqrt{5}+1)$ (E) none of these

59. A circle has a diameter with endpoints at $(-2,3)$ and $(6,5)$. Find its equation.

(A) $(x+2)^2 + (y-3)^2 = 68$

(B) $(x-4)^2 + (y-8)^2 = 61$

(C) $(x-2)^2 + (y-4)^2 = 17$

(D) $(x-2)^2 + (y-4)^2 = \sqrt{17}$

(E) none of these

60. Which figure represents the circle with equation

$$x^2 - 2x + y^2 + 6y = -6\ ?$$

Again, notice that there are four figures, but five choices.

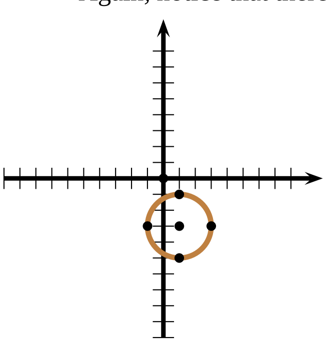


Figure D.29: A

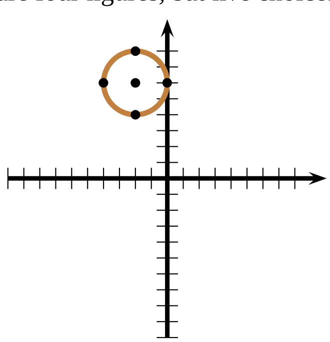


Figure D.30: B

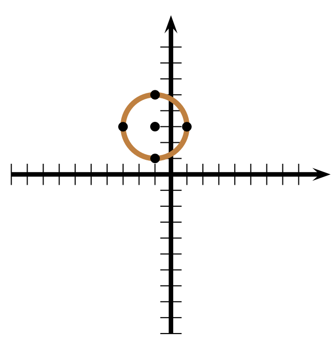


Figure D.31: C

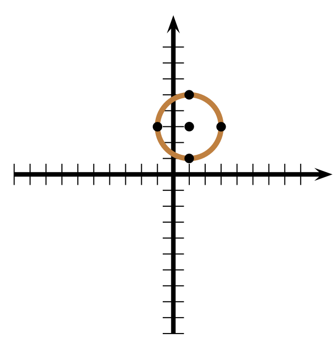


Figure D.32: D

(A) A (B) B (C) C (D) D (E) none of these

61. Which figure represents the semicircle with equation

$$x = 1 - \sqrt{-y^2 - 6y - 5}?$$

Again, notice that there are four figures, but five choices.

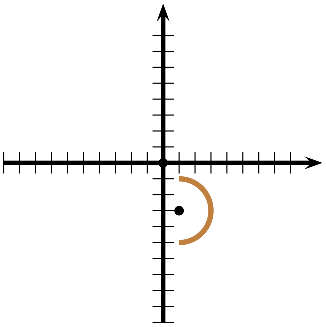


Figure D.33: A

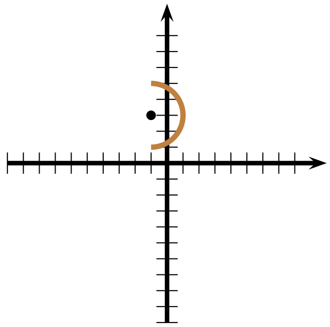


Figure D.34: B

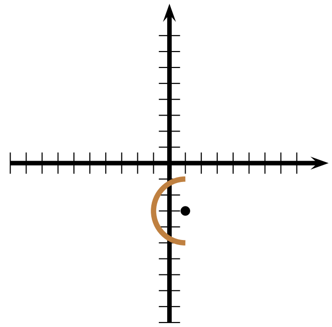


Figure D.35: C

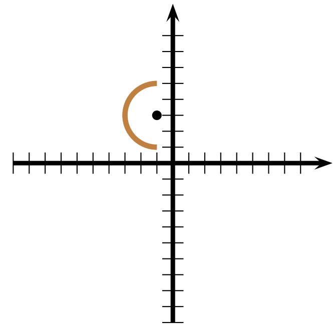


Figure D.36: D

(A) A (B) B (C) C (D) D (E) none of these

62. Find the equation of the circle with centre at $(-1,2)$ and passing through $(0,1)$.
(A) $(x-1)^2+(y+2)^2=10$
(B) $(x+1)^2+(y-2)^2=2$
(C) $(x+1)^2+(y-2)^2=10$
(D) $(x-1)^2+(y+2)^2=2$
(E) none of these

63. Let a and b be real constants. Find the centre and the radius of the circle with equation

$$x^2+2ax+y^2-4by=1.$$

(A) Centre: $(-a,2b)$, Radius: $\sqrt{a^2+4b^2}$
(B) Centre: $(a,2b)$, Radius: $\sqrt{1+a^2+4b^2}$
(C) Centre: $(a,-2b)$, Radius: $\sqrt{1+a^2+4b^2}$
(D) Centre: $(-a,2b)$, Radius: $\sqrt{1+a^2+4b^2}$
(E) none of these

64. A circle has a diameter with endpoints $A(b,-a)$ and $B(-b,a)$. Find its equation.
(A) $(x-b)^2+(y+a)^2=a^2+b^2$
(B) $(x-b)^2+(y-a)^2=a^2+b^2$
(C) $x^2+y^2=a^2+b^2$
(D) $x^2+y^2=\sqrt{a^2+b^2}$
(E) none of these

65. Find the centre C and the radius R of the circle with equation $x^2+y^2=2ax-b$.
(A) $C(0,0), R=\sqrt{2a-b}$
(B) $C\left(a,-\frac{b}{2}\right), R=\sqrt{a^2+\frac{b^2}{4}}$
(C) $C(-a,0), R=\sqrt{a^2-b}$
(D) $C(a,0), R=\sqrt{a^2-b}$
(E) none of these

D.1.8 Functions: Definition

66. Which one of the the following represents a function?

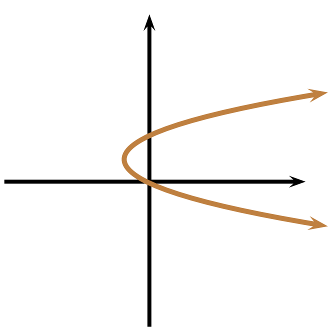


Figure D.37: A

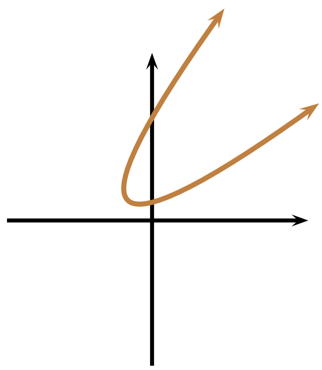


Figure D.38: B

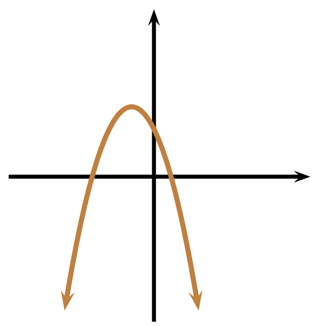


Figure D.39: C

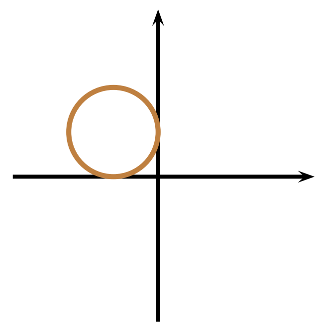


Figure D.40: D

(A) A (B) B (C) C (D) D (E) none of these

67. How many functions are there from the set $\{a,b,c\}$ to the set $\{1,2\}$?
(A) 9 (B) 8 (C) 6 (D) 1 (E) none of these

D.1.9 Evaluation of Formulæ

Figure D.41 shews a functional curve $y=f(x)$, and refers to problems 68 to 71.

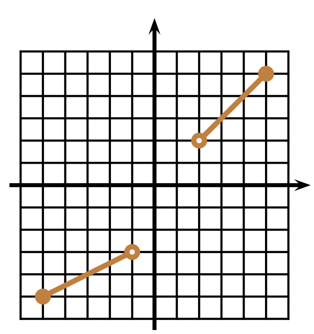


Figure D.41: Problems 68 to 71.

68. The domain of the functional curve in figure D.41 is
(A) $[-5;5]$ (B) $[-5;-1[\cup]2;5]$ (C) $[-5;-1]\cup[2;5]$ (D) $[-5;-1[\cup[2;5[$ (E) none of these

69. The image of the functional curve in figure D.41 is

(A) $[-5;5]$ (B) $[-5;-3]\cup]2;5]$ (C) $[-5;-3[\cup]2;5[$ (D) $[-5;-3[\cup]2;5[$ (E) none of these

70. $f(3) =$

(A) 1 (B) 2 (C) 3 (D) 5 (E) none of these

71. f is

(A) an even function (B) increasing (C) an odd function (D) decreasing (E) none of these

Problems 72 through 72 refer to the functional curve in figure D.42.

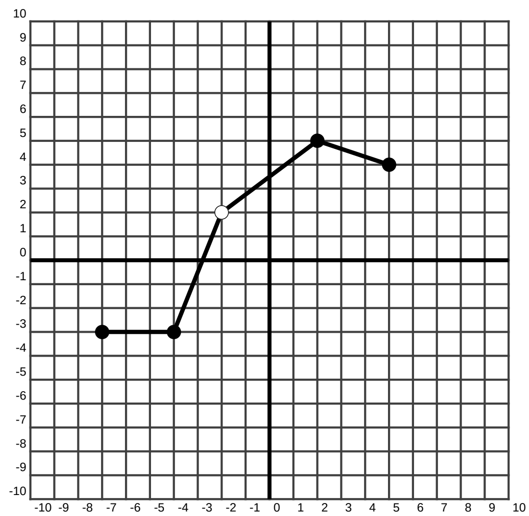


Figure D.42: Problems 72 through 72.

72. The domain of the function f is

(A) $[-7;5]$ (B) $[-7;-2[\cup]-2;5]$ (C) $]-7;5[$ (D) $]-7;5]$ (E) none of these

73. The image of the function f is

(A) $[-3;4]$ (B) $[-3;4]\setminus\{2\}$ (C) $[-3;5]$ (D) $[-3;2[\cup]2;5]$ ((E) none of these

74. $f(2) =$

(A) 2 (B) 3 (C) 4 (D) 5 (E) none of these

75. $f(-2) =$

(A) 2 (B) 3 (C) 5 (D) undefined (E) none of these

76. Let $f(x) = 1 + x + x^2$. What is $f(0) + f(1) + f(2)$?

(A) 10 (B) 11 (C) 7 (D) 3 (E) none of these

77. Let $f : \mathbb{R} \to \mathbb{R}$ with the assignment rule $x \mapsto (x - (x - (x-1)^2)^2)^2$. Find $f(2)$.

(A) 1 (B) 4 (C) 16 (D) 0 (E) none of these

78. Let $f(x) = \frac{x-1}{x+1}$. Find $f(2)$.

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{1}{2}$ (E) none of these

79. Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(\frac{x}{3}\right) = 9x$. Find $f(x)$.

(A) $3x$ (B) $\frac{x}{3}$ (C) $\frac{x}{9}$ (D) $27x$ (E) none of these

80. Consider $f(x) = \frac{1}{x}$, for $x \neq 0$. How many of the following assertions are necessarily true?

$$I: f(ab) = f(a)f(b), \quad II: f\left(\frac{a}{b}\right) = \frac{f(a)}{f(b)}, \quad III: f(a+b) = f(a)+f(b), \quad IV: f\left(\frac{1}{a}\right) = \frac{1}{f(a)}$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none of them

D.1.10 Algebra of Functions

81. Let $f(x) = 2x+1$. Find $(f \circ f \circ f)(1)$.

(A) 8 (B) 3 (C) 9 (D) 15 (E) none of these

82. Let $f(x) = x-2$ and $g(x) = 2x+1$. Find

$$(f \circ g)(1) + (g \circ f)(1).$$

(A) -1 (B) 1 (C) 0 (D) 2 (E) none of these

83. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(2x-1) = x+1$. Find $f(-3)$.

(A) -2 (B) 1 (C) -1 (D) 0 (E) none of these

84. Let $f(x) = x+1$. What is $\underbrace{(f \circ \cdots \circ f)}_{100\ f's}(x)$?

(A) $x+100$ (B) $x^{100}+1$ (C) $x^{100}+100$ (D) $x+99$ (E) none of these

85. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(1-x) = x-2$. Find $f(x)$.

(A) $-1-x$ (B) $x+1$ (C) $x-1$ (D) $1-x$ (E) none of these

Questions 86 through 90 refer to the assignment rules given by $f(x) = \frac{x}{x-1}$ and $g(x) = 1-x$.

86. Determine $(f \circ g)(2)$.

(A) 0 (B) -2 (C) -1 (D) $\frac{1}{2}$ (E) none of these

87. Determine $(g \circ f)(2)$.

(A) 0 (B) -2 (C) -1 (D) $\frac{1}{2}$ (E) none of these

88. Determine $(gf)(2)$.

(A) 0 (B) -2 (C) -1 (D) $\frac{1}{2}$ (E) none of these

89. Determine $(g+f)(2)$.

(A) 1 (B) -2 (C) -1 (D) $\frac{1}{2}$ (E) none of these

90. If $(f+g)(x) = (g \circ f)(x)$ then $x \in$

(A) $\{-1, 1\}$ (B) $\{-3, 0\}$ (C) $\{-3, 3\}$ (D) $\{0, 3\}$ (E) none of these

Problems 97 through 101 refer to the functions f and g with

$$f(x) = \frac{2}{2-x}, \qquad g(x) = \frac{x-2}{x-1}, \qquad h(x) = \frac{2x-2}{x}.$$

91. $f(-1) =$

(A) 4 (B) $\frac{2}{3}$ (C) 1 (D) $\frac{3}{2}$ (E) none of these

92. Find $(fgh)(-1)$.
(A) $\frac{37}{6}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) 4 (E) none of these

93. Find $(f+g+h)(-1)$.
(A) $\frac{37}{6}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) 4 (E) none of these

94. $(f \circ g)(x) =$
(A) $f(x)$ (B) $g(x)$ (C) $h(x)$ (D) x (E) none of these

95. $(g \circ h)(x) =$
(A) $f(x)$ (B) $g(x)$ (C) $h(x)$ (D) x (E) none of these

96. $(h \circ f)(x) =$
(A) $f(x)$ (B) $g(x)$ (C) $h(x)$ (D) x (E) none of these

Problems 97 through 101 refer to the functions f and g with $f(x) = \sqrt{x^2+1}$ and $g(x) = \sqrt{x^2-1}$.

97. Find $(fg)(2)$.
(A) 4 (B) 2 (C) $\sqrt{5}+\sqrt{3}$ (D) $\sqrt{15}$ (E) none of these

98. Find $(f+g)(2)$.
(A) 4 (B) 2 (C) $\sqrt{5}+\sqrt{3}$ (D) $\sqrt{15}$ (E) none of these

99. Find $(f \circ g)(2)$.
(A) 4 (B) 2 (C) $\sqrt{5}+\sqrt{3}$ (D) $\sqrt{15}$ (E) none of these

100. Find $(g \circ f)(2)$.
(A) 4 (B) 2 (C) $\sqrt{5}+\sqrt{3}$ (D) $\sqrt{15}$ (E) none of these

101. Find $(g \circ f \circ g \circ f \circ g \circ f \circ g \circ f)(2)$.
(A) 4 (B) 2 (C) $\sqrt{3}$ (D) $\sqrt{5}$ (E) none of these

102. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(2x) = x^2$. Find $(f \circ f)(x)$.
(A) x^4 (B) $\frac{x^4}{4}$ (C) $\frac{x^4}{16}$ (D) $\frac{x^4}{64}$ (E) none of these

D.1.11 Domain of Definition of a Formula

103. What is the natural domain of definition of the assignment rule $x \mapsto \frac{\sqrt{x^2-1}}{|x|-1}$?
(A) $[-1;1]$ (B) $]-\infty;-1] \cup [1;+\infty[$ (C) $]-\infty;-1[\cup]1;+\infty[$ (D) $\mathbb{R} \setminus \{\pm 1\}$ (E) none of these

104. What is the natural domain of definition of the assignment rule $x \mapsto \frac{\sqrt{x-2}}{x^3-8}$?
(A) $]2;+\infty[$ (B) $\mathbb{R} \setminus \{2\}$ (C) $]-\infty;-2[$ (D) $[2;+\infty[$ (E) none of these

Questions 105 through 108 are related.

105. Consider the assignment rule $x \mapsto \frac{1+x}{1-x}$. Find its domain of definition.

(A) $\mathbb{R} \setminus \{1\}$

(B) $[-1;1[$

(C) $\mathbb{R} \setminus \{-1,1\}$

(D) $\mathbb{R} \setminus \{-1\}$

(E) none of these

106. Consider the assignment rule $x \mapsto \sqrt{\frac{1+x}{1-x}}$. Find its domain of definition.

(A) $]-\infty;-1[\cup]1;+\infty[$

(B) $[-1;1[$

(C) $]-\infty;-1] \cup]1;+\infty[$

(D) $[-1;1]$

(E) none of these

107. Consider the assignment rule $x \mapsto \sqrt{1+x}+\sqrt{1-x}$. Find its domain of definition.

(A) $]-\infty;-1[\cup]1;+\infty[$

(B) $[-1;1[$

(C) $]-\infty;-1] \cup]1;+\infty[$

(D) $[-1;1]$

(E) none of these

108. Consider the assignment rule $x \mapsto \sqrt{\frac{1+x}{1-x}-1}$. Find its domain of definition.

(A) $]0;1[$

(B) $[0;1]$

(C) $[-1;1[$

(D) $[0;1[$

(E) none of these

109. What is the domain of definition of the formula $x \mapsto \sqrt{1-x^2}$?

(A) $[-1;1]$ (B) $]-\infty;-1]$ (C) $]-\infty;1]$ (D) $[1;+\infty[$ (E) none of these

110. Find the natural domain of definition of $x \mapsto \sqrt{-x}+\sqrt{1+x}$.

(A) $[-1;0]$ (B) $[0;1]$ (C) $[-1;1]$ (D) $\mathbb{R} \setminus [-1;1]$ (E) none of these

111. Find the natural domain of definition of $x \mapsto \sqrt{\frac{x}{x^2-x-6}}$.

(A) $[-2;3]$ (B) $[-2;0[\cup[3;+\infty[$ (C) $]-2;0]\cup]3;+\infty[$ (D) $]-3;+\infty[$ (E) none of these

D.1.12 Piecewise-defined Functions

112. Which one most resembles the graph of $y = f(x) = \begin{cases} \dfrac{1}{x}+1 & \text{if } x \in]-\infty;-1] \\ 1-x^2 & \text{if } x \in]-1;1[\\ \dfrac{1}{x}-1 & [1;+\infty[\end{cases}$?

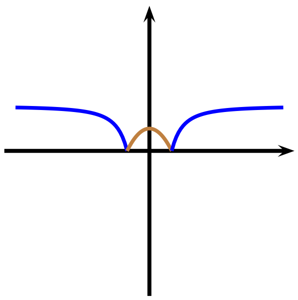


Figure D.43: A

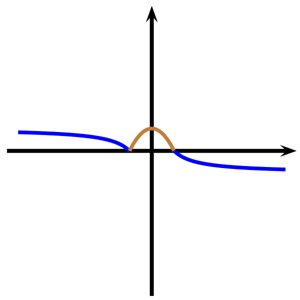


Figure D.44: B

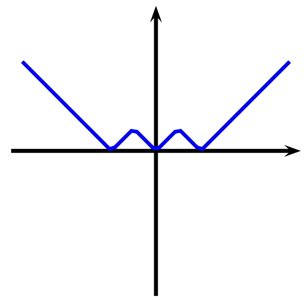


Figure D.45: C

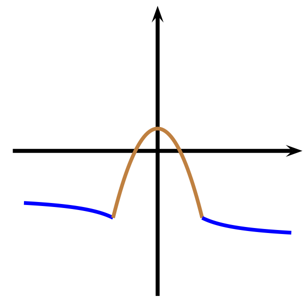


Figure D.46: D

(A) A (B) B (C) C (D) D (E) none of these

113. Which one most resembles the graph of $y = f(x) = \begin{cases} (x+3)^2-5 & \text{if } x \in]-\infty;-1] \\ x^3 & \text{if } x \in]-1;1[\\ 5-(x-3)^2 & [1;+\infty[\end{cases}$?

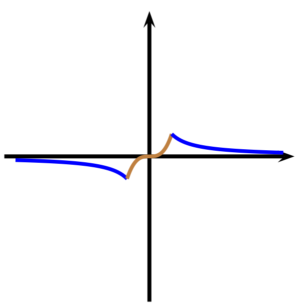


Figure D.47: A

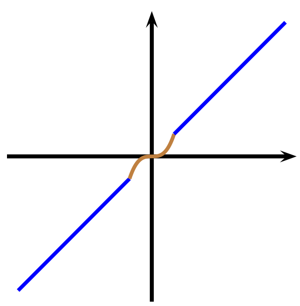


Figure D.48: B

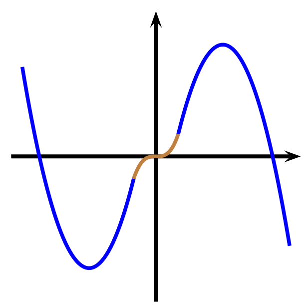


Figure D.49: C

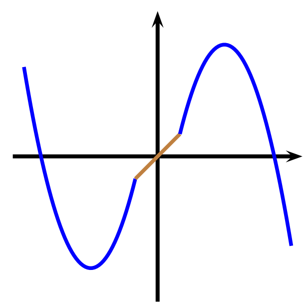


Figure D.50: D

(A) A (B) B (C) C (D) D (E) none of these

D.1.13 Parity of Functions

114. Which one of the following functions $f : \mathbb{R} \to \mathbb{R}$ with the assignment rules given below, represents an even function?
(A) $f(x) = x|x|$ (B) $f(x) = |x-x^2|$ (C) $f(x) = x^2 - x^4 + 1 - x$ (D) $f(x) = |x|^3$ (E) none of these

115. How many of the following are assignment rules of even functions?

$$I : a(x) = |x|^3, \qquad II : b(x) = x^2|x|, \qquad III : c(x) = x^3 - x, \qquad IV : d(x) = |x+1|$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none

116. Let f be an odd function and let g be an even function, both with the same domain. How many of the following functions are necessarily even?

$$I : x \mapsto f(x)g(x) \qquad II : x \mapsto f(x) + g(x) \qquad III : x \mapsto (f(x))^2 + (g(x))^2 \qquad IV : x \mapsto f(x)|g(x)|$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none of them

117. Let f be an even function and let g be an odd function, with $f(2) = 3$ and $g(2) = 5$. Find the value of $f(-2) + g(-2) + (fg)(-2)$.
(A) -17 (B) 23 (C) 13 (D) 7 (E) none of these

118. Let f be an even function and let g be an odd function, both defined over all reals. How many of the following functions are necessarily even?

$$I : x \mapsto (f+g)(x) \quad II : x \mapsto (f \circ g)(x) \quad III : x \mapsto (g \circ f)(x) \quad IV : x \mapsto |f(x)| + |g(x)|$$

(A) none (B) exactly one (C) exactly two (D) exactly three (E) all four

119. Let f be an odd function defined over all real numbers. How many of the following are necessarily even?

$$I : 2f; \qquad II : |f|; \qquad III : f^2; \qquad IV : f \circ f.$$

(A) Exactly one (B) Exactly two (C) Exactly three (D) All four (E) none is even

120. Let f be an odd function such that $f(-a) = b$ and let g be an even function such that $g(c) = a$. What is $(f \circ g)(-c)$?
(A) b (B) $-b$ (C) $-a$ (D) a (E) none of these

D.1.14 Transformations of Graphs

121. The curve $y = \dfrac{x-1}{x+1}$ experiences the following successive transformations: (1) a reflexion about the y axis, (2) a translation 1 unit down, (3) a reflexion about the x-axis. Find the equation of the resulting curve.
(A) $y = \dfrac{2}{1-x}$ (B) $y = \dfrac{x}{2-x}$ (C) $y = \dfrac{2}{x-1}$ (D) $y = \dfrac{x-2}{x}$ (E) none of these

122. What is the equation of the resulting curve after $y = x^2 - x$ has been, successively, translated one unit up and reflected about the y-axis?
(A) $y = x^2 - x + 1$ (B) $y = x^2 + x + 1$ (C) $y = -x^2 + x - 1$ (D) $y = (x+1)^2 - x - 1$ (E) none of these

123. What is the equation of the curve symmetric to the curve $y = \dfrac{1}{x^3} + 1$ with respect to the line $y = 0$?
(A) $y = -\dfrac{1}{x^3} + 1$ (B) $y = -\dfrac{1}{x^3} - 1$ (C) $y = \dfrac{1}{(x-1)^3}$ (D) $y = \dfrac{1}{(x-1)^{1/3}}$ (E) $y = \dfrac{1}{(1-x)^{1/3}}$

124. What is the equation of the resulting curve after the curve $y = x|x+1|$ has been successively translated one unit right and reflected about the y-axis?
(A) $y = (x-1)|x|$ (B) $y = -(x+1)|x|$ (C) $y = -x|x|$ (D) $y = -x|x| - 1$ (E) none of these

125. The curve $y = |x| + x$ undergoes the following successive transformations: (1) a translation 1 unit down, (2) a reflexion about the y-axis, (3) a translation 2 units right. Find the equation of the resulting curve.
(A) $y = |x-2| - x + 1$ (B) $y = |x-2| - x - 1$ (C) $y = |x+2| - x - 3$ (D) $y = |x-2| + x - 1$ (E) none of these

There are six graphs shewn below. The first graph is that of the original curve $y = f(x)$, and the other five are various transformations of the original graph. You are to match each graph letter below with the appropriate equation in 126 through 130 below.

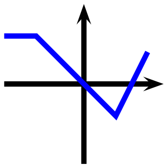


Figure D.51: $y = f(x)$.

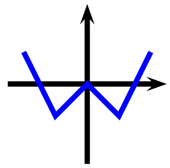


Figure D.52: A.

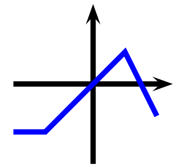


Figure D.53: B.

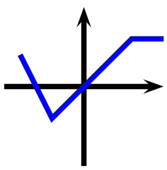


Figure D.54: C.

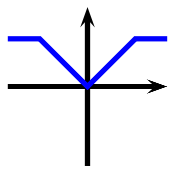


Figure D.55: D.

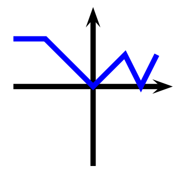


Figure D.56: E.

126. $y = f(-x)$ is
(A) (B) (C) (D) (E)

127. $y = -f(x)$ is
(A) (B) (C) (D) (E)

128. $y = f(|x|)$ is
(A) (B) (C) (D) (E)

129. $y = |f(x)|$ is
(A) (B) (C) (D) (E)

130. $y = f(-|x|)$ is
(A) (B) (C) (D) (E)

There are six graphs shewn below. The first graph is that of the original curve $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sqrt[3]{x}$, and the other five are various transformations of the original graph. You are to match each graph letter below with the appropriate equation in 131 through 135 below.

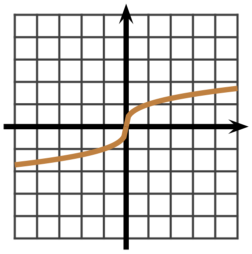


Figure D.57: $y = f(x)$.

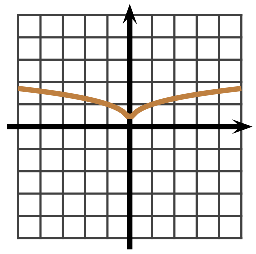


Figure D.58: A.

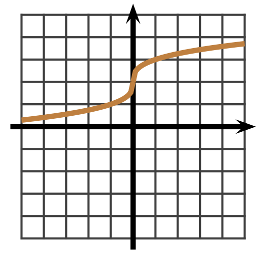


Figure D.59: B.

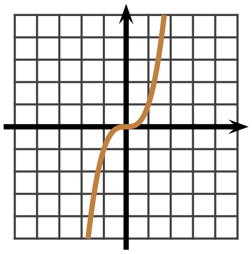


Figure D.60: C.

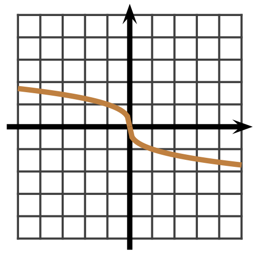


Figure D.61: D.

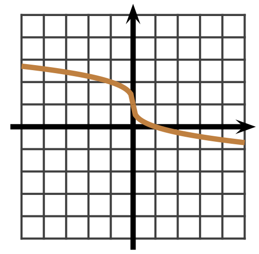


Figure D.62: E.

131. $y = f(x) + 1$ is
(A) (B) (C) (D) (E)

132. $y = f^{-1}(x)$ is
(A) (B) (C) (D) (E)

133. $y = -f(x) + 1$ is
(A) (B) (C) (D) (E)

134. $y = |f(x)|$ is
(A) (B) (C) (D) (E)

135. $y = f(-x)$ is
(A) (B) (C) (D) (E)

D.1.15 Quadratic Functions

136. Find the vertex of the parabola with equation $y = x^2 - 6x + 1$.
   (A) $(3,10)$ (B) $(-3,10)$ (C) $(-3,-8)$ (D) $(3,-8)$ (E) none of these

137. Find the equation of the parabola whose axis of symmetry is parallel to the y-axis, passes through $(2,1)$, and has vertex at $(-1,2)$.
   (A) $x = 3(y-2)^2 - 1$
   (B) $y = -9(x+1)^2 + 2$
   (C) $y = -(x-1)^2 + 2$
   (D) $y = -\frac{1}{9}(x+1)^2 + 2$
   (E) none of these

138. Let a, b, c be real constants. Find the vertex of the parabola $y = cx^2 + 2bx + a$.
   (A) $\left(-\frac{b}{2c}, a - \frac{3b^2}{4c}\right)$ (B) $\left(-\frac{b}{c}, a - \frac{b^2}{c}\right)$ (C) $\left(-\frac{b}{c}, a + \frac{b^2}{c}\right)$ (D) $\left(\frac{b}{c}, a + 3\frac{b^2}{c}\right)$ (E) none of these

139. A parabola has vertex at $(1,2)$, symmetry axis parallel to the x-axis, and passes through $(-1,0)$. Find its equation.
   (A) $x = -\frac{(y-2)^2}{2} + 1$
   (B) $x = -2(y-2)^2 + 1$
   (C) $y = -\frac{(x-1)^2}{2} + 2$
   (D) $y = -2(x-1)^2 + 2$
   (E) none of these

140. The graph in figure D.63 below belongs to a curve with equation of the form $y = A(x+1)^2 + 4$. Find A.

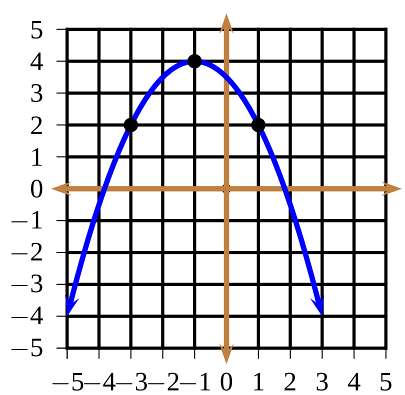


Figure D.63: Problem 144.

   (A) $A = \frac{1}{2}$ (B) $A = -1$ (C) $A = -\frac{1}{2}$ (D) $A = -2$ (E) none of these

Problems 141 through 143 refer to the quadratic function $q : \mathbb{R} \to \mathbb{R}$ with assignment rule given by

$$q(x) = x^2 - 6x + 5.$$

141. How many of the following assertions is (are) true?

   (a) q is convex.
   (b) q is invertible over $\mathbb{R}$.
   (c) the graph q has vertex $(-3, -4)$.
   (d) the graph of q has y-intercept $(0, 5)$ and x-intercepts $(-1, 0)$ and $(5, 0)$.

   (A) none (B) exactly one (C) exactly two (D) exactly three (E) all four

142. Which one most resembles the graph of q? Notice that there are four graphs but five choices.

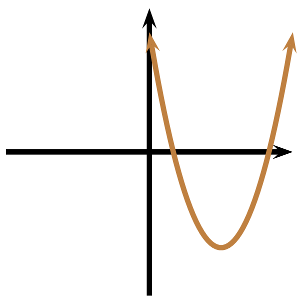


Figure D.64: A

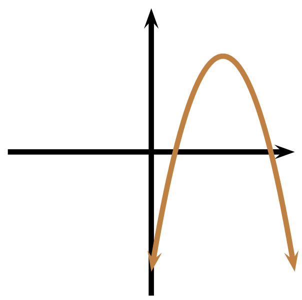


Figure D.65: B

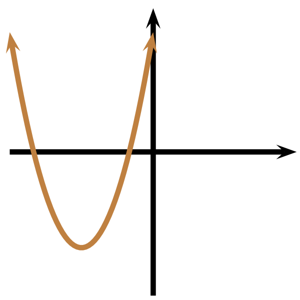


Figure D.66: C

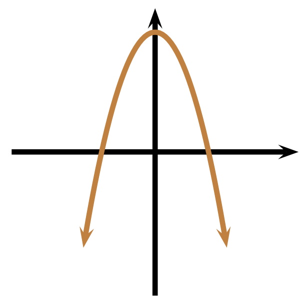


Figure D.67: D

   (A) A (B) B (C) C (D) D (E) none of these

143. Which one most resembles the graph of $y = q(|x|)$? Notice that there are four graphs but five choices.

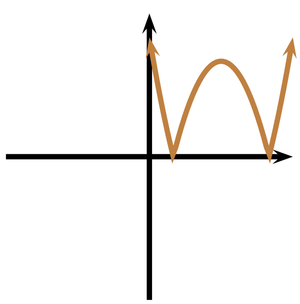


Figure D.68: A

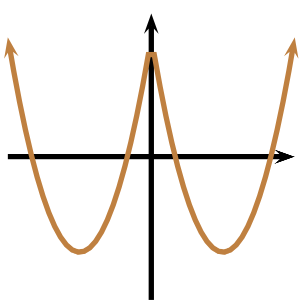


Figure D.69: B

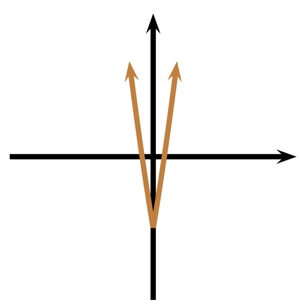


Figure D.70: C

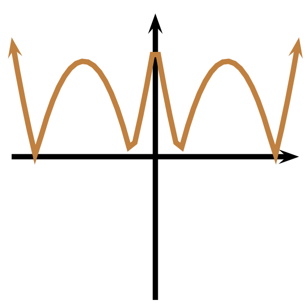


Figure D.71: D

   (A) A (B) B (C) C (D) D (E) none of these

144. Find the equation of the parabola shewn below. You may assume that the points marked with a dot have integer coordinates.

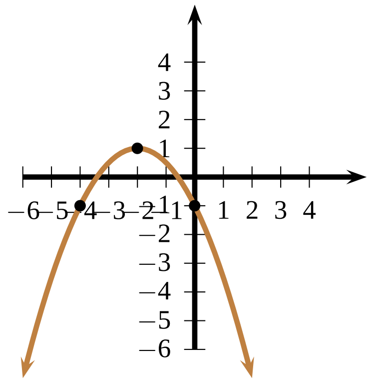


Figure D.72: Problem 144.

(A) $y = -\dfrac{-(x+2)^2}{2} + 1$

(B) $y = -2(x+2)^2 + 1$

(C) $y = (x+2)^2 + 1$

(D) $y = -(x+2)^2 + 1$

(E) none of these

D.1.16 Injections and Surjections

145. How many injective functions are there from the set $\{a, b, c\}$ to the set $\{1, 2\}$?

(A) 6 (B) 9 (C) 8 (D) 0 (E) none of these

146. How many surjective functions are there from the set $\{a, b, c\}$ to the set $\{1, 2\}$?

(A) 0 (B) 6 (C) 9 (D) 8 (E) none of these

147. How many invertible functions are there from the set $\{a, b, c\}$ to the set $\{1, 2\}$?

(A) 0 (B) 6 (C) 9 (D) 8 (E) none of these

D.1.17 Inversion of Functions

148. What is the equation of the curve symmetric to the curve $y = \dfrac{1}{x^3} + 1$ with respect to the line $y = x$?

(A) $y = -\dfrac{1}{x^3} + 1$ (B) $y = -\dfrac{1}{x^3} - 1$ (C) $y = \dfrac{1}{(x-1)^3}$ (D) $y = \dfrac{1}{(x-1)^{1/3}}$ (E) $y = \dfrac{1}{(1-x)^{1/3}}$

Figure D.73 shews a functional curve

$$f : [-5;5] \to [-3;6], \qquad y = f(x),$$

and refers to problems 149 to 153.

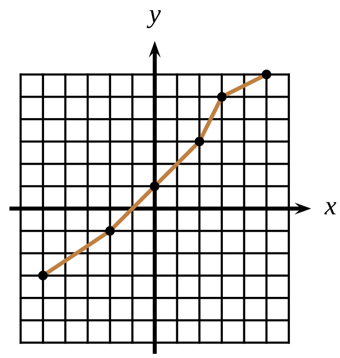


Figure D.73: Problems 149 to 153.

149. $f(-2) + f(2) =$

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

150. $f(-3)$ belongs to the interval

(A) $[-1;0]$ (B) $[-2;-1]$ (C) $[-3;-2]$ (D) $[0;1]$ (E) none of these

151. $f^{-1}(3) =$

(A) -3 (B) $-\dfrac{1}{3}$ (C) 2 (D) 5 (E) none of these

152. $(f \circ f)(2) =$

(A) 4 (B) 5 (C) 6 (D) undefined (E) none of these

153. The graph of f^{-1} is

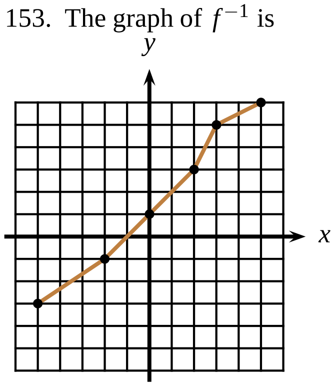


Figure D.74: A

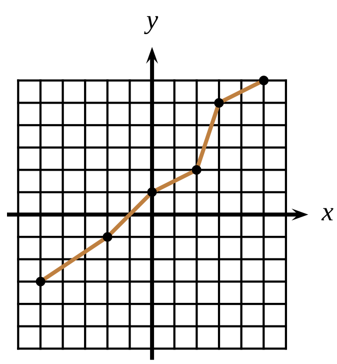


Figure D.75: B

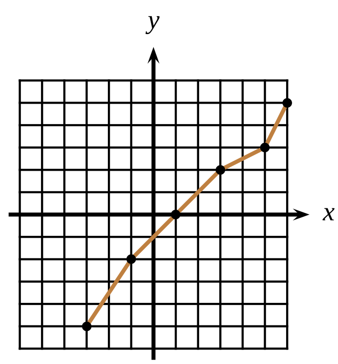


Figure D.76: C

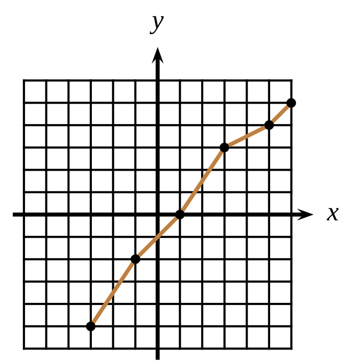


Figure D.77: D

(A) A (B) B (C) C (D) D (E) none of these

154. Let $f(x) = \dfrac{x}{x+1}$. Find $g(x)$ such that $(f \circ g)(x) = x$.

(A) $g(x) = \dfrac{x}{x-1}$ (B) $g(x) = \dfrac{x}{1+x}$ (C) $g(x) = \dfrac{x}{1-x}$ (D) $g(x) = -\dfrac{x}{1+x}$ (E) none of these

155. Let $f(x) = \dfrac{x+1}{1-2x}$. Then $f^{-1}(x) =$

(A) $\dfrac{1-x}{1+2x}$ (B) $\dfrac{1+x}{1-2x}$ (C) $\dfrac{x-1}{1+2x}$ (D) $\dfrac{1-x}{1-2x}$ (E) none of these

Problems 156 through 159 refer to the function f with assignment rule

$$y = f(x) = \begin{cases} \dfrac{x}{3} - \dfrac{10}{3} & \text{if } x \in [-5;-2[\\ 2x & \text{if } x \in [-2;2] \\ \dfrac{x}{3} + \dfrac{10}{3} & \text{if } x \in]2;5] \end{cases}$$

156. Which one most resembles the graph of f?

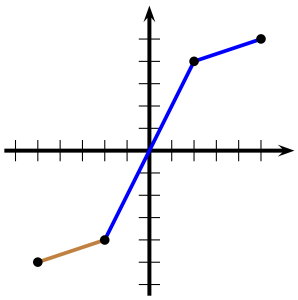


Figure D.78: A

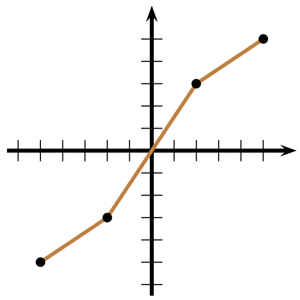


Figure D.79: B

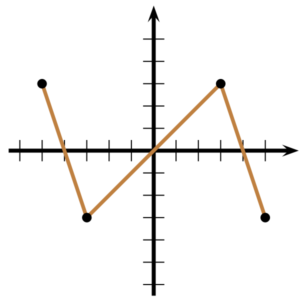


Figure D.80: C

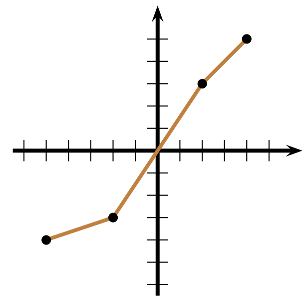


Figure D.81: D

(A) A (B) B (C) C (D) D (E) none of these

157. Find the exact value of $(f \circ f)(2)$.

(A) 4 (B) $\dfrac{14}{3}$ (C) 8 (D) 3 (E) none of these

158. Which one could not possibly be a possible value for $\underbrace{(f \circ \cdots \circ f)}_{n \text{ compositions}}(a)$, where n is a positive integer and $a \in [-5;5]$?.

(A) 0 (B) -5 (C) 5 (D) 6 (E) none of these

159. Which one most resembles the graph of f^{-1}?

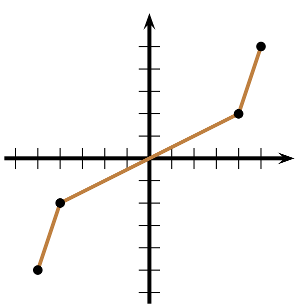


Figure D.82: A

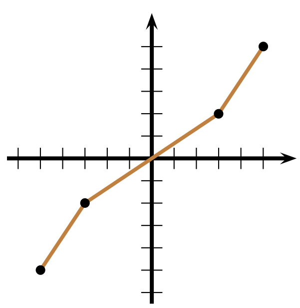


Figure D.83: B

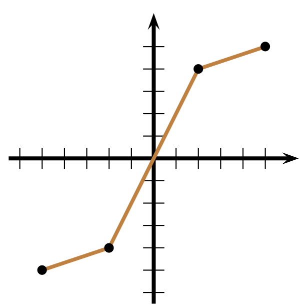


Figure D.84: C

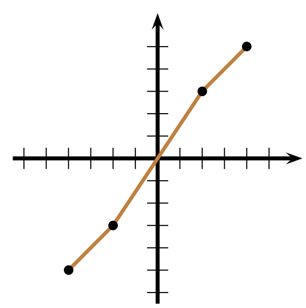


Figure D.85: D

(A) A (B) B (C) C (D) D (E) none of these

160. Let $f(x) = x - 2$ and $g(x) = 2x + 1$. Find $(f^{-1} \circ g^{-1})(x)$.

(A) $\dfrac{x+1}{2}$ (B) $\dfrac{x+3}{2}$ (C) $2x - 3$ (D) $2x - 1$ (E) none of these

161. Which of the following graphs represents an invertible function?

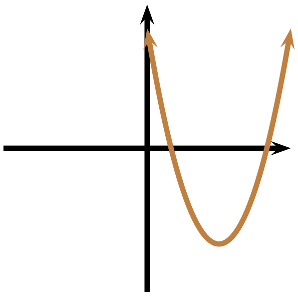


Figure D.86: A

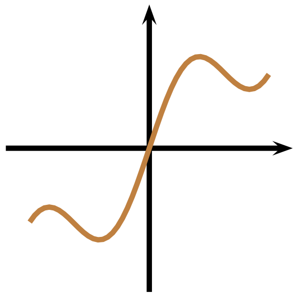


Figure D.87: B

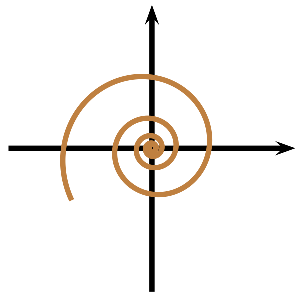


Figure D.88: C

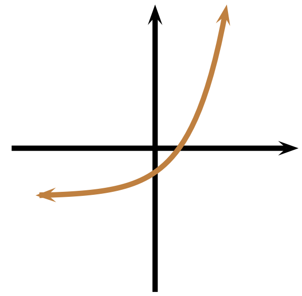


Figure D.89: D

(A) A (B) B (C) C (D) D (E) none of these

162. Let $f(x) = \left(\frac{x}{3} - 1\right)^3 + 2$. Then $f^{-1}(x) =$

(A) $3\sqrt[3]{x+2} - 3$ (B) $3\sqrt[3]{x-2} - 3$ (C) $3\sqrt[3]{x-2} + 3$ (D) $3\sqrt[3]{x+2} + 3$ (E) none of these

163. Let $f(x) = \dfrac{2x}{x+1}$. Find $f^{-1}(x)$.

(A) $\dfrac{x+1}{2x}$ (B) $\dfrac{x}{x-2}$ (C) $\dfrac{x-2}{x}$ (D) $\dfrac{x}{2-x}$ (E) none of these

164. Let $f(x) = (x+1)^5 - 2$. Find $f^{-1}(x)$.

(A) $\sqrt[5]{x+1} - 2$ (B) $\sqrt[5]{x-2} + 1$ (C) $\dfrac{1}{(x+1)^5 - 2}$ (D) $\sqrt[5]{x+2} - 1$ (E) none of these

165. Let $f(x) = -\dfrac{x}{2} + 1$. Find $f^{-1}(x)$.

(A) $\dfrac{2}{x} - 1$ (B) $-2x - 1$ (C) $2x - 1$ (D) $-2x + 2$ (E) none of these

166. Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = 1 - x$. Determine $(g \circ f)^{-1}(x)$.

(A) $\dfrac{x-1}{x}$ (B) $\dfrac{1-x}{x}$ (C) $\dfrac{1}{x-1}$ (D) $\dfrac{1}{1-x}$ (E) none of these

167. Let $f(x) = \dfrac{x+1}{x}$. Determine $f^{-1}(x)$.

(A) $f^{-1}(x) = \dfrac{x}{x-1}$

(B) $f^{-1}(x) = \dfrac{1}{x+1}$

(C) $f^{-1}(x) = \dfrac{1}{x-1}$

(D) $f^{-1}(x) = \dfrac{x}{x+1}$

(E) none of these

D.1.18 Polynomial Functions

168. Let p be a polynomial of degree 3 with roots at $x = 1$, $x = -1$, and $x = 2$. If $p(0) = 4$, find $p(4)$.
(A) 0 (B) 4 (C) 30 (D) 60 (E) none of these

169. A polynomial of degree 3 satisfies $p(0) = 0$, $p(1) = 0$, $p(2) = 0$, and $p(3) = -6$. What is $p(4)$?
(A) 0 (B) 1 (C) -24 (D) 24 (E) none of these

170. Factor the polynomial $x^3 - x^2 - 4x + 4$.
(A) $(x+1)(x-2)(x+2)$
(B) $(x-1)(x+1)(x-4)$
(C) $(x-1)(x-2)(x+2)$
(D) $(x-1)(x+1)(x+4)$
(E) none of these

171. Determine the value of the parameter a so that the polynomial $x^3 + 2x^2 + ax - 10$ be divisible by $x - 2$.
(A) $a = 3$ (B) $a = -3$ (C) $a = -2$ (D) $a = -1$ (E) none of these

172. A polynomial leaves remainder -1 when divided by $x - 2$ and remainder 2 when divided by $x + 1$. What is its remainder when divided by $x^2 - x - 2$?
(A) $x - 1$ (B) $2x - 1$ (C) $-x - 1$ (D) $-x + 1$ (E) none of these

Questions 173 through 176 refer to the polynomial p in figure D.90. The polynomial has degree 5. You may assume that the points marked with dots have integer coordinates.

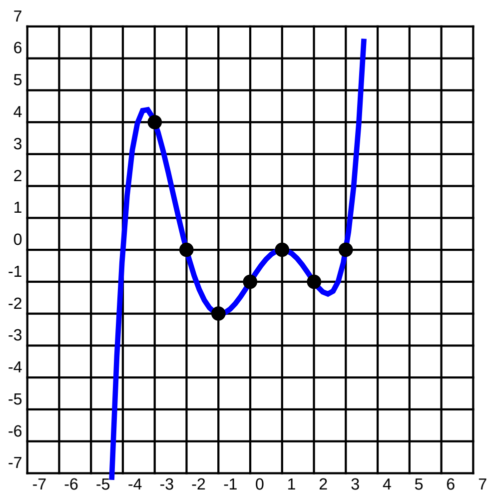


Figure D.90: Problems 173 through 176.

173. Determine the value of $p(0)$.
(A) 0 (B) -1 (C) 4 (D) -2 (E) none of these

174. Determine the value of $p(-3)$.
(A) 0 (B) -1 (C) 4 (D) -2 (E) none of these

175. Determine $p(x)$.

(A) $\dfrac{(x-3)(x+2)(x+4)(x-1)^2}{24}$

(B) $(x-3)(x+2)(x+4)(x-1)^2$

(C) $\dfrac{(x-3)(x+2)(x+4)(x-1)}{24}$

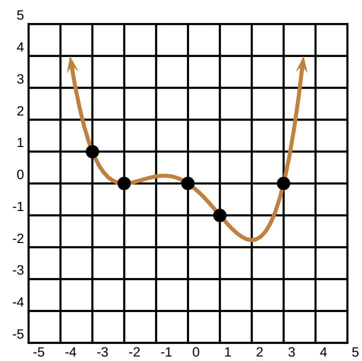
(D) $(x-3)(x+2)(x+4)(x-1)$

(E) none of these

176. Determine the value of $(p \circ p)(-3)$.

(A) 4 (B) 18 (C) 20 (D) 24 (E) none of these

177. The polynomial p whose graph is shewn below has degree 4. You may assume that the points marked below with a dot through which the polynomial passes have have integer coordinates. Find its equation.



(A) $p(x) = x(x+2)^2(x-3)$

(B) $p(x) = -\dfrac{x(x+2)^2(x-3)}{18}$

(C) $p(x) = \dfrac{x(x+2)^2(x-3)}{12}$

(D) $p(x) = \dfrac{x(x+2)^2(x-3)}{18}$

(E) none of these

Problems 178 through 180 refer to the polynomial in figure D.91, which has degree 4. You may assume that the points marked below with a dot through which the polynomial passes have have integer coordinates.

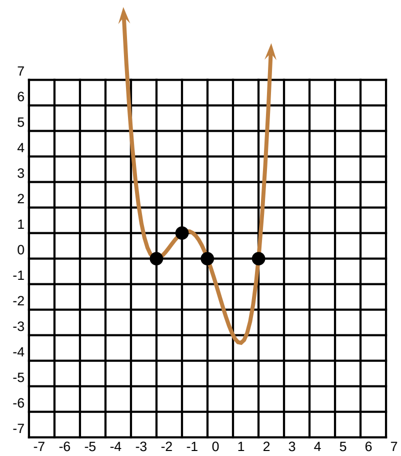


Figure D.91: Problems 178through 180.

178. Determine $p(-1)$.

(A) 1 (B) -1 (C) 3 (D) -3 (E) none of these

179. $p(x) =$

(A) $x(x+2)^2(x-2)$

(B) $\dfrac{x(x-2)^2(x+2)}{3}$

(C) $\dfrac{x(x+2)^2(x-2)}{3}$

(D) $x(x+2)(x-2)^2$

(E) none of these

180. Determine $(p \circ p)(-1)$.

(A) 1 (B) 3 (C) -3 (D) -1 (E) none of these

D.1.19 Rational Functions

181. Which graph most resembles the curve $y = \dfrac{1}{x-1} + 2$?

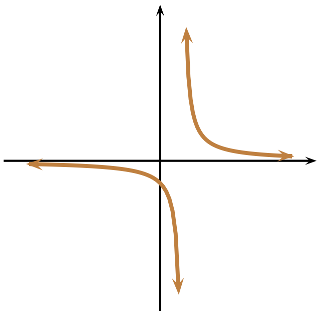


Figure D.92: A

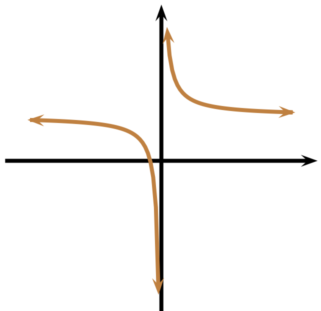


Figure D.93: B

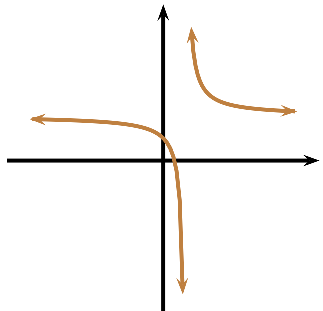


Figure D.94: C

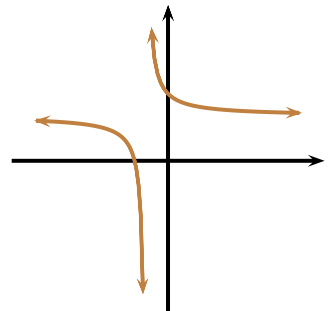


Figure D.95: D

(A) (B) (C) (D) (E) none of these

182. Which graph most resembles the curve $y = \left|\dfrac{1}{x-1} + 2\right|$?

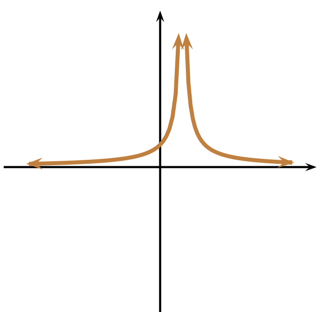


Figure D.96: A

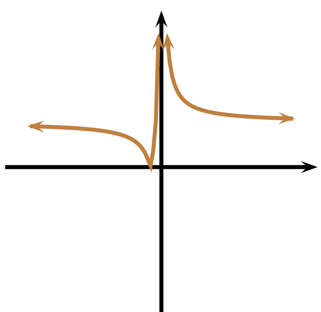


Figure D.97: B

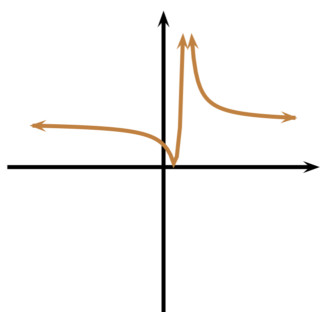


Figure D.98: C

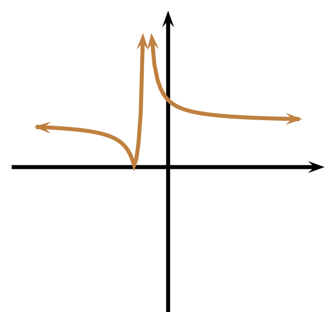


Figure D.99: D

(A) (B) (C) (D) (E) none of these

183. Which graph most resembles the curve $y = \dfrac{1}{|x|-1} + 2$?

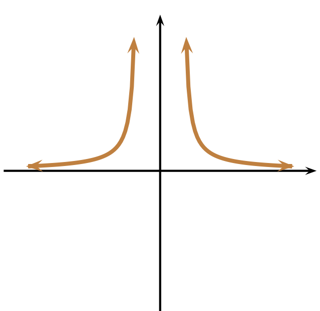


Figure D.100: A

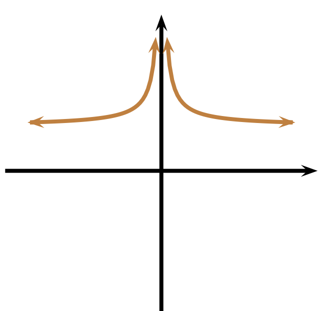


Figure D.101: B

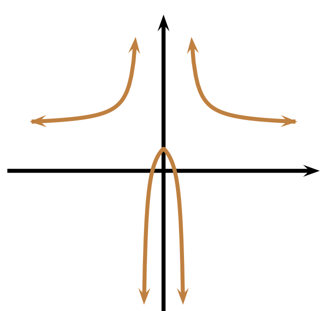


Figure D.102: C

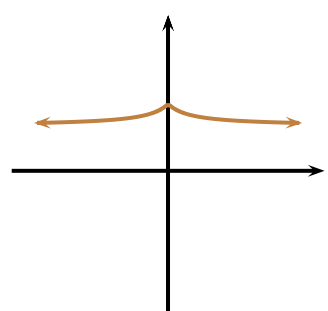


Figure D.103: D

(A) (B) (C) (D) (E) none of these

Situation: Problems 184 through 188 refer to the rational function f, with $f(x) = \dfrac{x^2+x}{x^2+x-2}$.

184. As $x \to +\infty$, $y \to$

(A) $+\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$ (C) 0 (D) 1 (E) none of these

185. The y-intercept of f is located at

(A) $(0,-1)$ (B) $(0,\frac{1}{2})$ (C) $(0,1)$ (D) $(0,0)$ (E) none of these

186. Which of the following is true?

(A) f has zeroes at $x=0$ and $x=-1$, and poles at $x=1$ and $x=-2$.

(B) f has zeroes at $x=0$ and $x=1$, and poles at $x=1$ and $x=2$.

(C) f has zeroes at $x=0$ and $x=-1$, and poles at $x=-1$ and $x=2$.

(D) f has no zeroes and no poles

(E) none of these

187. Which of the following is the sign diagram for f?

(A)

$]-\infty;-2[$	$]-2;-1[$	$]-1;0[$	$]0;1[$	$]1;+\infty[$
+	−	+	−	−

(B)

$]-\infty;-2[$	$]-2;-1[$	$]-1;0[$	$]0;1[$	$]1;+\infty[$
−	−	+	+	+

(C)

$]-\infty;-2[$	$]-2;-1[$	$]-1;0[$	$]0;1[$	$]1;+\infty[$
+	+	−	+	−

(D)

$]-\infty;-2[$	$]-2;-1[$	$]-1;0[$	$]0;1[$	$]1;+\infty[$
+	−	+	−	+

(E) none of these

188. The graph of $y = f(x)$ most resembles

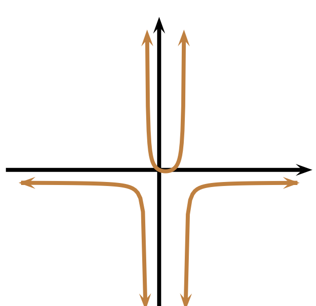


Figure D.104: A

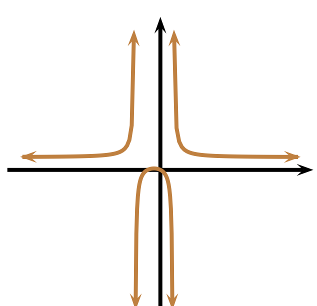


Figure D.105: B

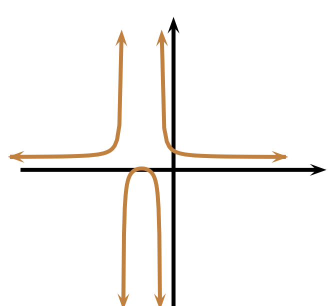


Figure D.106: C

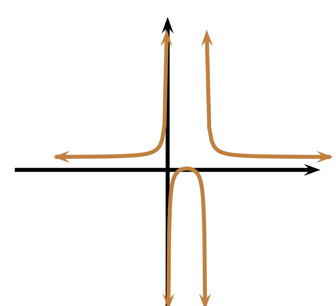


Figure D.107: D

(A) (B) (C) (D) (E) none of these

Situation: Problems **??** through 193 refer to the rational function f, with $f(x) = \dfrac{(x+1)^2(x-2)}{(x-1)(x+2)^2}$.

189. As $x \to +\infty$, $y \to$

(A) $+\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 0 (D) 1 (E) none of these

190. The y-intercept of f is located at

(A) $(0,-1)$ (B) $(0,1)$ (C) $(0,-\frac{1}{2})$ (D) $(0,\frac{1}{2})$ (E) none of these

191. Which of the following is true?

(A) f has zeroes at $x=-1$ and $x=2$, and poles at $x=1$ and $x=-2$.

(B) f has zeroes at $x=1$ and $x=-2$, and poles at $x=-1$ and $x=2$.

(C) f has zeroes at $x=1$ and $x=2$, and poles at $x=-1$ and $x=-2$.

(D) f has no zeroes and no poles

(E) none of these

192. Which of the following is the sign diagram for f?

(A)

$]-\infty;-2[$	$]-2;-1[$	$]-1;1[$	$]1;2[$	$]2;+\infty[$
+	−	+	−	−

(B)

$]-\infty;-2[$	$]-2;-1[$	$]-1;1[$	$]1;2[$	$]2;+\infty[$
−	−	+	+	+

(C)

$]-\infty;-2[$	$]-2;-1[$	$]-1;1[$	$]1;2[$	$]2;+\infty[$
+	+	−	+	−

(D)

$]-\infty;-2[$	$]-2;-1[$	$]-1;1[$	$]1;2[$	$]2;+\infty[$
+	+	+	−	+

(E) none of these

193. The graph of $y = f(x)$ most resembles

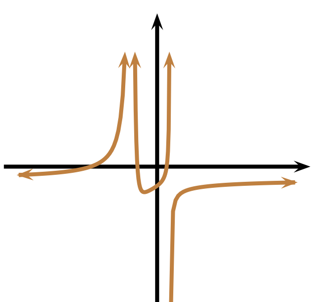


Figure D.108: A

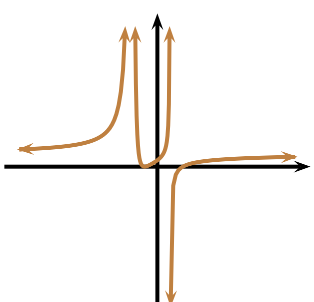


Figure D.109: B

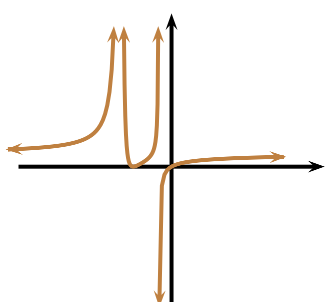


Figure D.110: C

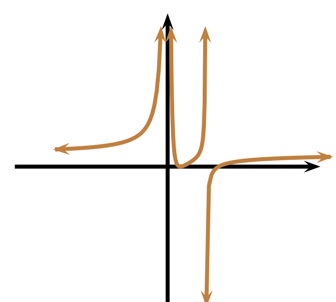


Figure D.111: D

(A) (B) (C) (D) (E) none of these

Situation: Problems 194 through 196 refer to the rational function f whose graph appears in figure **??**. The function f is of the form

$$f(x) = K\frac{(x-a)(x-b)^2}{(x-c)^4},$$

where K, a, b, c are real constants that you must find. It is known that $f(x) \to +\infty$ as $x \to 1$.

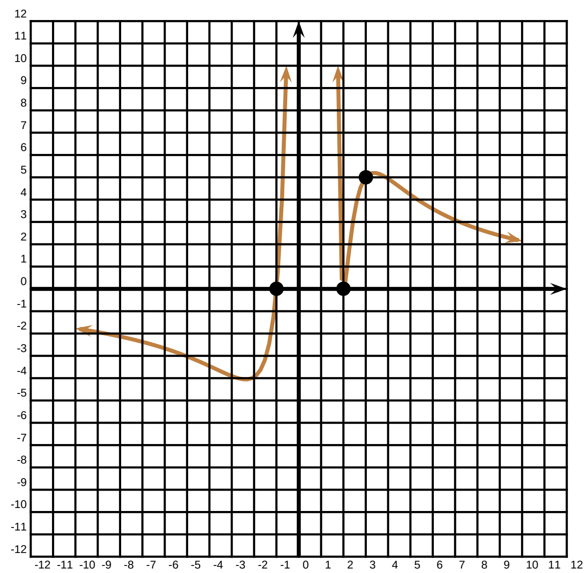


Figure D.112: Problems **??** through **??**.

194. Which of the following is true?
   (A) $a = 1, b = -1, c = 2$
   (B) $a = -1, b = 2, c = 1$
   (C) $a = -1, b = 1, c = 2$
   (D) $a = 2, b = -1, c = 1$
   (E) none of these

195. What is the value of K?

(A) 10 (B) 20 (C) -20 (D) 1 (E) none of these

196. As $x \to +\infty$, $f(x) \to$

(A) 0 (B) 1 (C) $+\infty$ (D) $-\infty$ (E) none of these

Situation: Problems 197 through 201 refer to the rational function f, with $f(x) = \dfrac{x^3}{x^2-4}$.

197. As $x \to +\infty$, $y \to$

(A) $+\infty$ (B) $-\infty$ (C) 0 (D) 1 (E) none of these

198. As $x \to -\infty$, $y \to$

(A) $+\infty$ (B) $-\infty$ (C) 0 (D) 1 (E) none of these

199. Where are the poles of f?

(A) $x = 2$ and $x = -2$ (B) $x = -1$ and $x = -2$ (C) $x = 0$ and $x = 2$ (D) $x = 0$ and $x = -2$ (E) none of these

200. Which of the following is true?

(A) $x = 0$ is the only zero of f

(B) $x = -2$ and $x = +2$ are the only zeroes of f

(C) $x = 0$, $x = 2$, and $x = -2$ are all zeroes of f

(D) f has no zeroes

(E) none of these

201. The graph of $y = f(x)$ most resembles

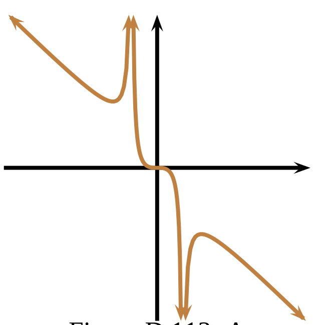


Figure D.113: A

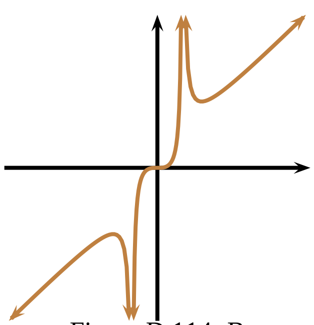


Figure D.114: B

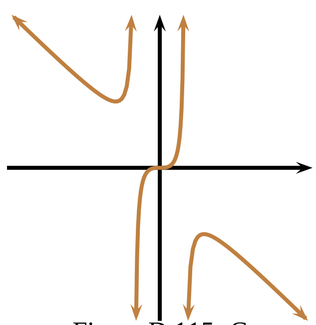


Figure D.115: C

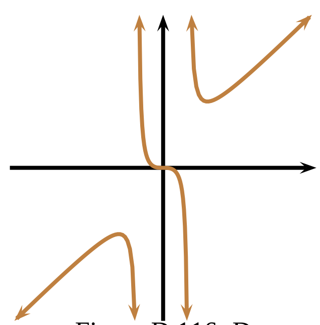


Figure D.116: D

(A) (B) (C) (D) (E) none of these

Situation: Problems 202 through 206 refer to the rational function f, with $f(x) = \dfrac{(x-1)(x+2)}{(x+1)(x-2)}$.

202. Which of the following is a horizontal asymptote for f?

(A) $y = -1$ (B) $y = 1$ (C) $y = 0$ (D) $y = 2$ (E) none of these

203. Where are the poles of f?

(A) $x = 1$ and $x = -2$ (B) $x = -1$ and $x = -2$ (C) $x = -1$ and $x = 2$ (D) $x = 1$ and $x = 2$ (E) none of these

204. Where are the zeroes of f?

(A) $x = 1$ and $x = -2$ (B) $x = -1$ and $x = -2$ (C) $x = -1$ and $x = 2$ (D) $x = 1$ and $x = 2$ (E) none of these

205. What is the y-intercept of f?

(A) $(0,1)$ (B) $(0,2)$ (C) $(0,-1)$ (D) $(0,-2)$ (E) none of these

206. The graph of $y = f(x)$ most resembles

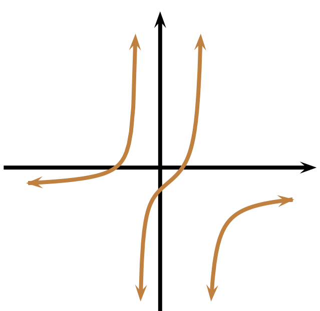


Figure D.117: A

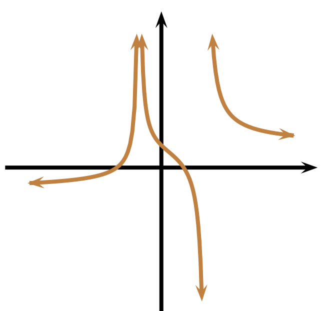


Figure D.118: B

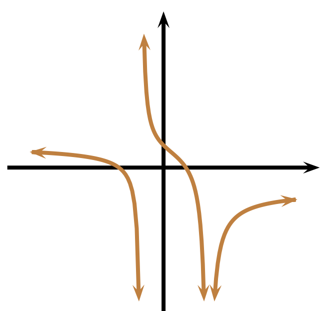


Figure D.119: C

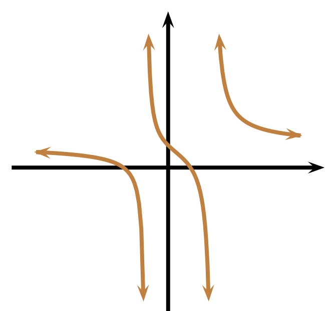


Figure D.120: D

(A) (B) (C) (D) (E) none of these

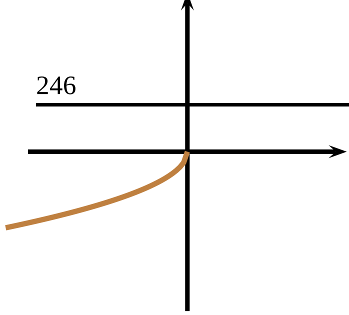


Figure D.122: A

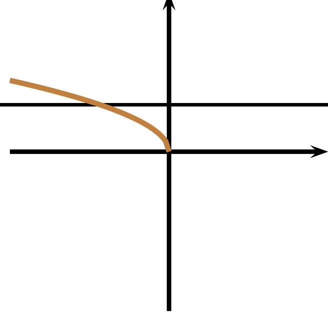


Figure D.123: B

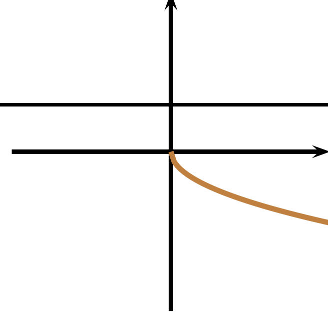


Figure D.124: C

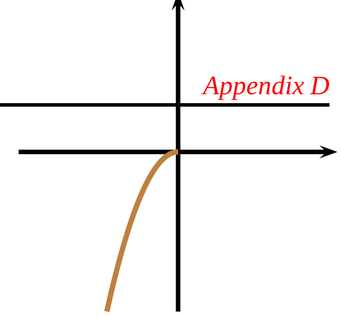


Figure D.125: D

D.1.20 Algebraic Functions

207. The graph in figure D.121 below belongs to a curve with equation of the form $y = A\sqrt{x+3} - 2$. Find A.

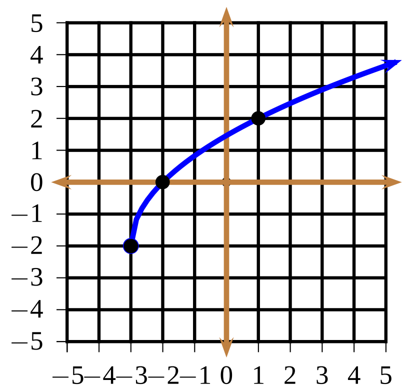


Figure D.121: Problem 207.

(A) $A = \frac{1}{2}$ (B) $A = 1$ (C) $A = -2$ (D) $A = 2$ (E) none of these

208. Which one of the following graphs best represents the curve $y = -\sqrt{-x}$?

(A) A (B) B (C) C (D) D (E) none of these

209. Which graph most resembles the curve $y = -\sqrt{x-1}$?

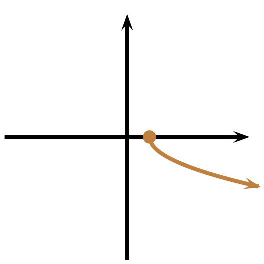


Figure D.126: A

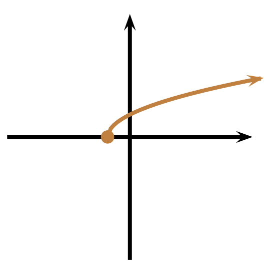


Figure D.127: B

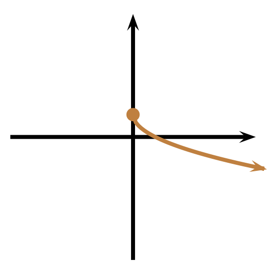


Figure D.128: C

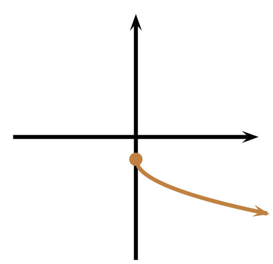


Figure D.129: D

(A) (B) (C) (D) (E) none of these

210. Which graph most resembles the curve $y = \sqrt{1-x}$?

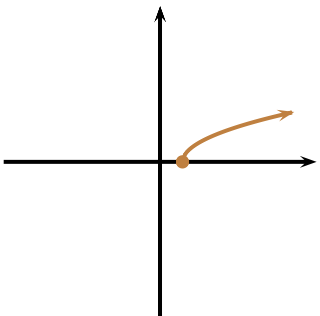


Figure D.130: A

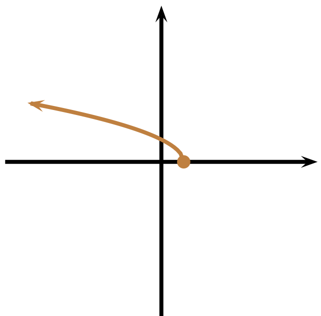


Figure D.131: B

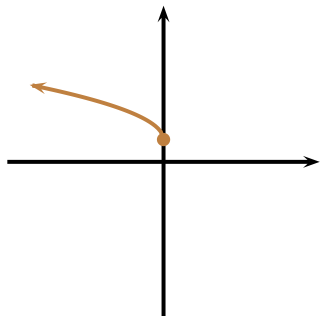


Figure D.132: C

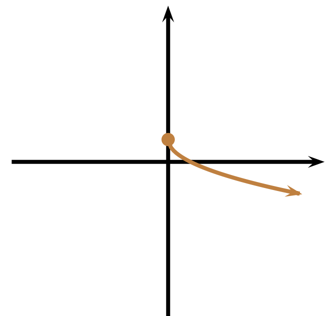


Figure D.133: D

(A) (B) (C) (D) (E) none of these

Situation: Problems 211 through 214 refer to the assignment rule given by $a(x) = \sqrt{\dfrac{x+1}{x-1}}$.

211. What is the domain of definition of a?
(A) $[-1;1[$ (B) $[-1;1]$ (C) $]-\infty;-1]\cup[1;+\infty[$ (D) $]-\infty;-1]\cup]1;+\infty[$ (E) none of these

212. What is $a(2)$?
(A) $\sqrt{3}$ (B) $\dfrac{1}{\sqrt{3}}$ (C) $\sqrt{2}$ (D) undefined (E) none of these

213. $a^{-1}(x) =$
(A) $\dfrac{1-x^2}{1+x^2}$ (B) $\left(\dfrac{1+x}{1-x}\right)^2$ (C) $\dfrac{1+x^2}{1-x^2}$ (D) $\dfrac{1+x^2}{x^2-1}$ (E) none of these

214. The graph of a most resembles

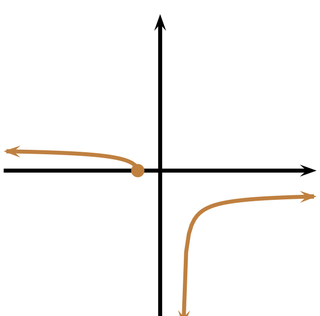


Figure D.134: A

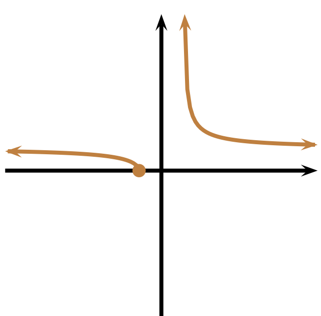


Figure D.135: B

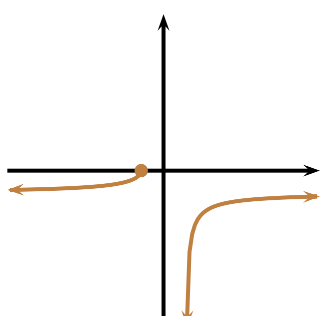


Figure D.136: C

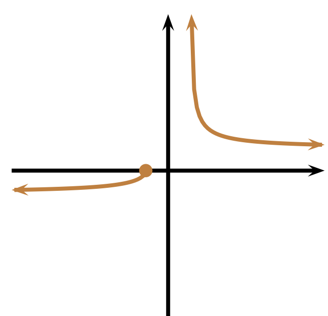


Figure D.137: D

(A) (B) (C) (D) (E) none of these

D.1.21 Conics

215. Find the equation of the ellipse in figure D.138.

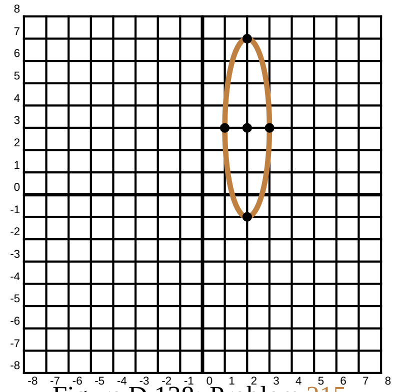


Figure D.138: Problem 215.

(A) $(x-2)^2+\dfrac{(y-3)^2}{16}=1$

(B) $(x+2)^2+\dfrac{(y+3)^2}{16}=1$

(C) $(x-2)^2+\dfrac{(y-3)^2}{4}=1$

(D) $(x+2)^2+\dfrac{(y+3)^2}{4}=1$

(E) none of these

216. Find the equation of the hyperbola in figure D.139.

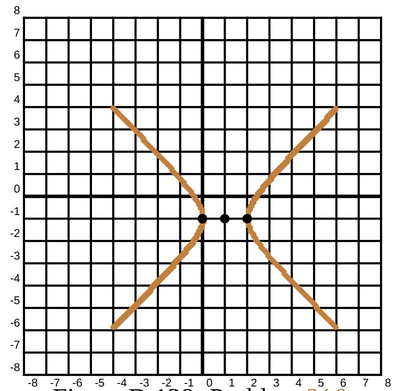


Figure D.139: Problem 216.

(A) $(x-1)^2-(y-1)^2=1$

(B) $(x-1)^2-(y+1)^2=1$

(C) $(y-1)^2-(x-1)^2=1$

(D) $(y+1)^2-(x-1)^2=1$

(E) none of these

D.1.22 Geometric Series

217. Find the sum of the terms of the infinite geometric progression

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots.$$

(A) $\frac{4}{3}$ (B) $\frac{3}{4}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) none of these

D.1.23 Exponential Functions

218. Which of the following best resembles the graph of the curve $y=2^{-|x|}$?

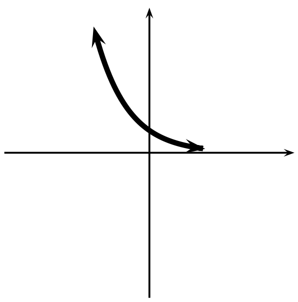


Figure D.140: A

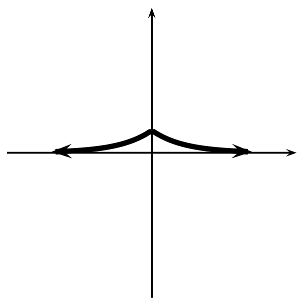


Figure D.141: B

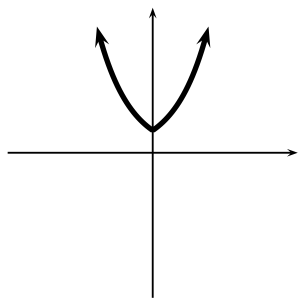


Figure D.142: C

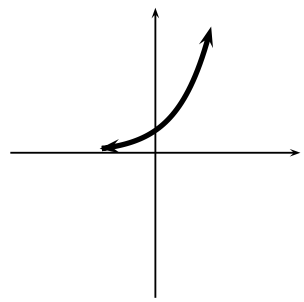


Figure D.143: D

(A) A (B) B (C) C (D) D (E) none of these

219. If $3^{x^2}=81$, then

(A) $x\in\{-4,4\}$ (B) $x\in\{-9,9\}$ (C) $x\in\{-2,2\}$ (D) $x\in\{-3,-3\}$ (E) none of these

220. If the number 5^{2000} is written out (in decimal notation), how many digits does it have?

(A) 1397 (B) 1398 (C) 1396 (D) 2000 (E) none of these

D.1.24 Logarithmic Functions

221. Which of the following best resembles the graph of the curve $y = \log_{1/2} x$?

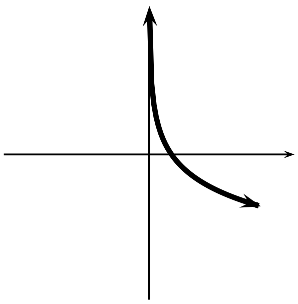


Figure D.144: A

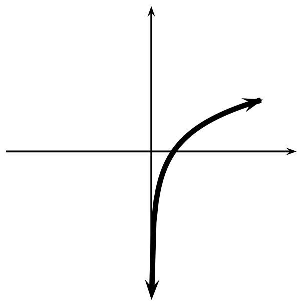


Figure D.145: B

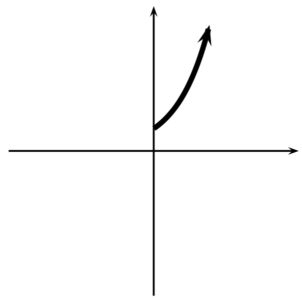


Figure D.146: C

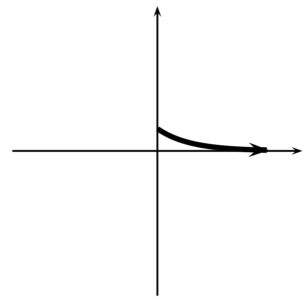


Figure D.147: D

(A) A (B) B (C) C (D) D (E) none of these

222. Find the smallest integer n for which the inequality $2^n > 4n^2 + n$ will be true.
(A) $n = 4$ (B) $n = 7$ (C) $n = 8$ (D) $n = 9$ (E) none of these

223. Solve the equation $9^x + 3^x - 6 = 0$.
(A) $x \in \{1, \log_3 2\}$ (B) $x \in \{\log_3 2\}$ only (C) $x \in \{1\}$ only (D) $x \in \{\log_2 3, \log_3 2\}$ (E) none of these

224. Find the exact value of $\log_{3\sqrt{3}} 729$.
(A) $\frac{1}{9}$ (B) $\frac{1}{4}$ (C) 9 (D) 4 (E) none of these

225. Let a and b be consecutive integers such that $a < \log_5 100 < b$. Then
(A) $a = 1; b = 2$ (B) $a = 2; b = 3$ (C) $a = 3; b = 4$ (D) $a = 4; b = 5$ (E) none of these

226. Find all real solutions to the equation $\log_2 \log_3 \log_2 x = 1$.
(A) $x = 512$ (B) $x = 81$ (C) $x = 256$ (D) $x = 12$ (E) none of these

227. Which of the following functions is (are) increasing in its (their) domain of definition?

$$I : x \mapsto \frac{1}{2^x}; \qquad II : x \mapsto 2^x; \qquad III : x \mapsto \log_{1/2} x.$$

(A) I and III only (B) II only (C) II and III only (D) III only (E) none of these

228. Which of the following assertions is (are) true for all strictly positive real numbers x and y?

$$I : \log_2 x + \log_2 y = \log_2(x+y); \qquad II : (\log_2 x)(\log_2 y) = \log_4 xy; \qquad III : 2^{\log_2 x} = x.$$

(A) I and III only (B) II only (C) II and III only (D) III only (E) none of these

229. $\log_8 2 =$
(A) $\frac{1}{4}$ (B) 3 (C) $\frac{1}{3}$ (D) 4 (E) none of these

230. $\log_2 8 =$
(A) 2 (B) 3 (C) 4 (D) 5 (E) none of these

231. $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8) =$
(A) 2 (B) 3 (C) 4 (D) 5 (E) none of these

232. If $\log_x 5 = 2$ then

(A) $x \in \{-\sqrt{5}, \sqrt{5}\}$ (B) $x \in \{\sqrt{5}\}$ only (C) $x \in \{2\}$ only (D) $x \in \{1, 2\}$ (E) none of these

233. If $\log_x 2x = 2$ then

(A) $x \in \{0, 2\}$ (B) $x \in \{0\}$ only (C) $x \in \{2\}$ only (D) $x \in \{1, 2\}$ (E) none of these

234. Given that $a > 1$, $t > 0$, $s > 0$ and that

$$\log_a t^3 = p, \qquad \log_{\sqrt{a}} s^2 = q,$$

find $\log_a st$ in terms of p and q.

(A) $\frac{p}{3} + \frac{q}{2}$ (B) $\frac{p}{3} + \frac{q}{4}$ (C) $3p + 4q$ (D) $\frac{p}{3} + q$ (E) none of these

235. Given that $a > 1$, $s > 1$, $t > 1$, and that

$$\log_a \sqrt{t} = p, \qquad \log_s a^2 = 2p^2,$$

find $\log_s t$ in terms of p.

(A) p^3 (B) $\frac{2}{p^3}$ (C) $2p^3$ (D) $\frac{p^2}{2}$ (E) none of these

236. What is the domain of definition of

$$x \mapsto \log_x (1 - x^2)?$$

(A) $[-1;1]$ (B) $]0;1]$ (C) $]0;1[$ (D) $]-1;1[$ (E) none of these

D.1.25 Goniometric Functions

237. How many solutions does $1 - \cos 2x = \frac{1}{2}$ have in the closed interval $[-\frac{\pi}{2}; \pi]$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

238. How many of the following assertions are true for all real numbers x?

$$I : \csc^2 x + \sec^2 x = 1; \qquad II : |\csc x| \geq 1; \qquad III : |\arcsin x| \leq 1; \qquad IV : \sin(2\pi + x) = \sin x$$

(A) none (B) exactly one (C) exactly two (D) exactly three (E) all four

239. Which of the following is a solution to the equation $\cos(2x - 1) = \frac{1}{2}$?

(A) $\frac{\pi}{6} + \frac{1}{2}$ (B) $\frac{\pi}{3} + \frac{1}{2}$ (C) $\frac{\pi}{6} - \frac{1}{2}$ (D) $\frac{\pi}{3} - \frac{1}{2}$ (E) none of these

240. If $\tan\theta = \frac{1}{4}$ and $\mathscr{C}\ \theta$ is in the third quadrant, find $\sin\theta$.

(A) $\frac{-\sqrt{17}}{4}$ (B) $-\frac{4}{\sqrt{17}}$ (C) $-\frac{1}{\sqrt{17}}$ (D) $\frac{1}{\sqrt{17}}$ (E) none of these

241. Find $\arcsin(\sin 10)$.

(A) 10 (B) $10 - 3\pi$ (C) $3\pi - 10$ (D) $10 - \frac{7\pi}{2}$ (E) none of these

242. Find $\sin(\arcsin 4)$.

(A) 4 (B) $\sqrt{15}$ (C) $\sqrt{17}$ (D) $4 - \pi$ (E) not a real number

243. $\sec^2 x + \csc^2 x =$

(A) $(\sec^2 x)(\csc^2 x)$ (B) $(\sec x)(\csc x)$ (C) $\sec x + \csc x$ (D) $\tan^2 x + \cot^2 x$ (E) none of these

Situation: Let $\sin x = \frac{1}{3}$ and $\sin y = \frac{1}{4}$ where x and y are acute angles. Problems 244 through 249 refer to this situation.

244. Find $\cos x$.

(A) $\dfrac{2}{3}$ (B) $\dfrac{2\sqrt{2}}{3}$ (C) $-\dfrac{2}{3}$ (D) $-\dfrac{2\sqrt{2}}{3}$ (E) none of these

245. Find $\cos 2x$.

(A) $\dfrac{2}{3}$ (B) $\dfrac{4\sqrt{2}}{3}$ (C) $\dfrac{7}{9}$ (D) $\dfrac{\sqrt{2}}{3}$ (E) none of these

246. Find $|\cos\frac{x}{2}|$.

(A) $\dfrac{1}{3}$ (B) $\frac{1}{2}\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{3}}$ (C) $\sqrt{\dfrac{17}{18}}$ (D) $\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{3}}$ (E) none of these

247. Find $\cos y$.

(A) $\dfrac{3}{4}$ (B) $\dfrac{\sqrt{15}}{4}$ (C) $-\dfrac{3}{4}$ (D) $-\dfrac{\sqrt{15}}{4}$ (E) none of these

248. Find $\sin(x+y)$.

(A) $\dfrac{7}{12}$ (B) $\dfrac{1}{12}$ (C) $\dfrac{2\sqrt{2}}{9}+\dfrac{\sqrt{15}}{16}$ (D) $\dfrac{\sqrt{15}+2\sqrt{2}}{12}$ (E) none of these

249. Find $\cos(x+y)$.

(A) $\dfrac{\sqrt{30}}{6}+\dfrac{1}{12}$ (B) $\dfrac{\sqrt{30}}{12}-\dfrac{1}{12}$ (C) $\dfrac{\sqrt{30}}{12}+\dfrac{1}{12}$ (D) $\dfrac{\sqrt{30}}{6}-\dfrac{1}{12}$ (E) none of these

250. Which of the following is a real number solution to $2^{\cos x} = 3$?

(A) $\arccos\left(\dfrac{\ln 2}{\ln 3}\right)$ (B) $\arccos\left(\ln\dfrac{3}{2}\right)$ (C) $\arccos\left(\dfrac{\ln 3}{\ln 2}\right)$ (D) $\arccos(\ln 6)$ (E) there are no real solutions

251. $(\cos 2x)(\cos\frac{x}{2}) =$

(A) $\frac{1}{2}\sin\frac{5}{2}x - \frac{1}{2}\sin\frac{3}{2}x$

(B) $\frac{1}{2}\sin\frac{5}{2}x + \frac{1}{2}\sin\frac{3}{2}x$

(C) $\frac{1}{2}\cos\frac{5}{2}x + \frac{1}{2}\cos\frac{3}{2}x$

(D) $\frac{1}{2}\cos\frac{5}{2}x - \frac{1}{2}\cos\frac{3}{2}x$

(E) none of these

252. It is known that $\cos\frac{2\pi}{5} = \dfrac{\sqrt{5}-1}{2}$. Find $\cos\frac{\pi}{5}$.

(A) $\dfrac{\sqrt{5}-1}{4}$ (B) $\dfrac{\sqrt{5}+1}{2}$ (C) $-\dfrac{\sqrt{1+\sqrt{5}}}{2}$ (D) $\dfrac{\sqrt{1+\sqrt{5}}}{2}$ (E) none of these

253. It is known that $\cos\frac{2\pi}{5} = \dfrac{\sqrt{5}-1}{2}$. Find $\cos\frac{4\pi}{5}$.

(A) $2-\sqrt{5}$ (B) $\sqrt{5}-2$ (C) $3-\sqrt{5}$ (D) $\dfrac{3-\sqrt{5}}{2}$ (E) none of these

254. Find the smallest positive solution to the equation $\cos x^2 = 0$.

(A) 0 (B) $\dfrac{\sqrt{2\pi}}{2}$ (C) $\dfrac{\sqrt{\pi}}{2}$ (D) $\dfrac{\pi}{2}$ (E) none of these

255. $\cos\frac{223\pi}{6} =$

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $-\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{3}}{2}$ (E) none of these

256. If $2\cos^2 x + \cos x - 1 = 0$ and $x \in [0;\pi]$ then

(A) $x \in \left\{\frac{\pi}{3}, \pi\right\}$ (B) $x \in \left\{\frac{\pi}{2}, \pi\right\}$ (C) $x \in \left\{\frac{\pi}{3}, \frac{\pi}{4}\right\}$ (D) $x \in \left\{\frac{\pi}{3}, \frac{\pi}{6}\right\}$ (E) none of these

257. If $2\sin^2 x - \cos x - 1 = 0$ and $x \in [0;\pi]$ then

(A) $x \in \left\{\frac{\pi}{3}, \pi\right\}$ (B) $x \in \left\{\frac{\pi}{2}, \pi\right\}$ (C) $x \in \left\{\frac{\pi}{3}, \frac{\pi}{4}\right\}$ (D) $x \in \left\{\frac{\pi}{3}, \frac{\pi}{6}\right\}$ (E) none of these

D.1.26 Trigonometry

Situation: Questions 258 through 262 refer to the following. Assume that α and β are acute angles. Assume also that $\tan\alpha = \frac{1}{3}$ and that $\sec\beta = 3$.

258. Find $\sin\alpha$.

(A) $\frac{1}{4}$ (B) $\frac{3\sqrt{10}}{10}$ (C) $\frac{\sqrt{10}}{30}$ (D) $\frac{\sqrt{10}}{10}$ (E) none of these

259. Find $\sin\beta$.

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{10}}{3}$ (C) $\frac{2\sqrt{2}}{3}$ (D) $\frac{\sqrt{2}}{3}$ (E) none of these

260. Find $\cos\alpha$.

(A) $\frac{1}{4}$ (B) $\frac{3\sqrt{10}}{10}$ (C) $\frac{\sqrt{10}}{30}$ (D) $\frac{\sqrt{10}}{10}$ (E) none of these

261. Find $\cos\beta$.

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{10}}{3}$ (C) $\frac{2\sqrt{2}}{3}$ (D) $\frac{\sqrt{2}}{3}$ (E) none of these

262. Find $\cos(\alpha+\beta)$.

(A) $\frac{\sqrt{10}}{10} - \frac{2\sqrt{5}}{15}$ (B) $\frac{\sqrt{10}}{10} + \frac{2\sqrt{5}}{15}$ (C) $\frac{2\sqrt{5}}{5} - \frac{\sqrt{10}}{30}$ (D) $\frac{2\sqrt{5}}{5} + \frac{\sqrt{10}}{30}$ (E) none of these

Situation: Questions 263 through 268 refer to the following. $\triangle ABC$ is right-angled at A, $a = 4$ and $\sec B = 4$. Assume standard labelling.

263. Find $\sin C$.

(A) $\frac{1}{4}$ (B) $\frac{3\sqrt{15}}{15}$ (C) $\frac{\sqrt{15}}{4}$ (D) $\frac{4\sqrt{15}}{15}$ (E) none of these

264. Find $\angle C$, in radians.

(A) $\arcsin\frac{1}{4}$ (B) $\arccos\frac{1}{4}$ (C) $\arcsin\frac{\sqrt{15}}{4}$ (D) $\arccos\frac{4}{\sqrt{15}}$ (E) none of these

265. Find b.

(A) 1 (B) $\sqrt{15}$ (C) 4 (D) 16 (E) none of these

266. Find R, the radius of the circumscribed circle to $\triangle ABC$.

(A) 2 (B) $\frac{\sqrt{15}}{2}$ (C) $2\sqrt{15}$ (D) $\sqrt{15}$ (E) none of these

267. Find the area of $\triangle ABC$.

(A) 2 (B) $\frac{\sqrt{15}}{2}$ (C) $2\sqrt{15}$ (D) $\sqrt{15}$ (E) none of these

268. Find r, the radius of the inscribed circle to $\triangle ABC$.

(A) $\frac{\sqrt{15}}{2\sqrt{15}+10}$ (B) $\frac{\sqrt{15}}{\sqrt{15}+5}$ (C) $\frac{\sqrt{15}+5}{\sqrt{15}}$ (D) 2 (E) none of these

D.2 Old Exam Match Questions

Match the equation with the appropriate graph. Observe that there are fewer graphs than equations, hence, some blank spaces will remain blank.

1. $x - y^2 = 3$, ________
2. $x^2 - y^2 = 9$, ________
3. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, ________
4. $y^2 - x^2 = 9$, ________
5. $x^2 + y = 3$, ________
6. $x + y^2 = 3$, ________
7. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, ________
8. $x^2 + y^2 = 9$, ________
9. $y - x^2 = 3$, ________
10. $x + y = 3$, ________

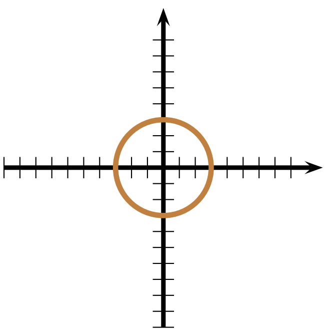


Figure D.148: Allan

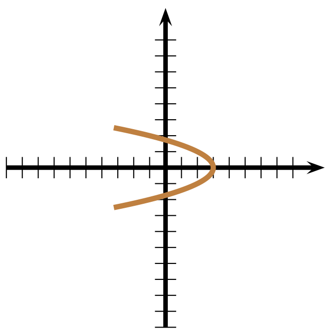


Figure D.149: Bob

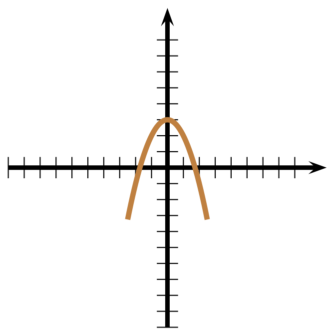


Figure D.150: Carmen

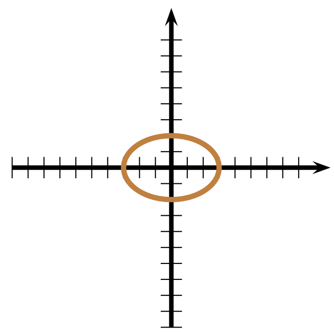


Figure D.151: Donald

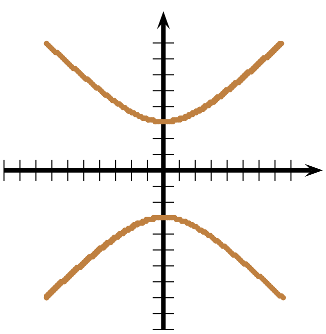


Figure D.152: Edgard

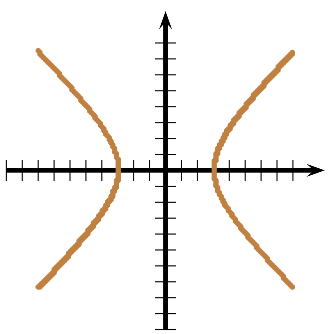


Figure D.153: Frances

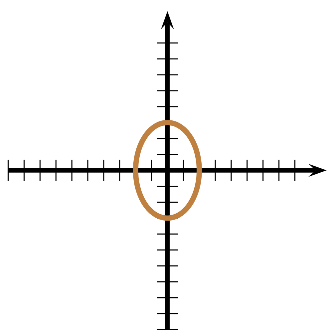


Figure D.154: Gertrude

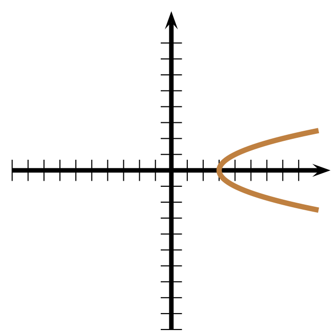


Figure D.155: Harry

Figure D.156 shows a functional curve $y = f(x)$. You are to match the letters of figures D.157 to D.167 with the equations on α through μ below. Some figures may not match with any equation, or viceversa.

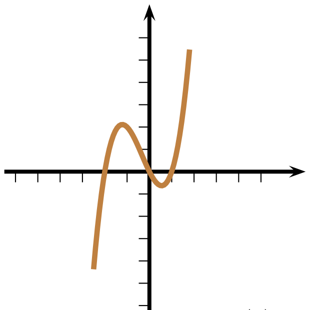


Figure D.156: $y = f(x)$

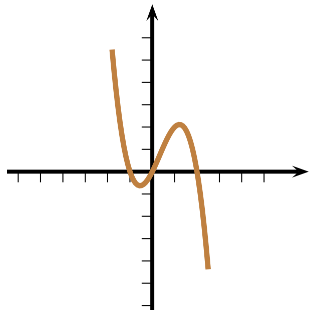


Figure D.157: A

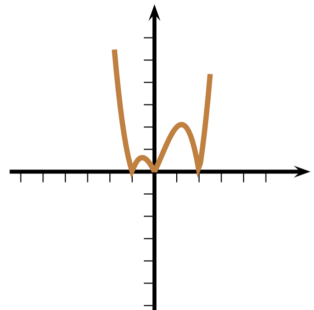


Figure D.158: B

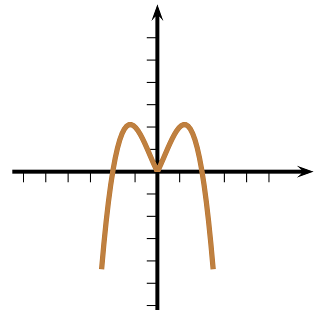


Figure D.159: C

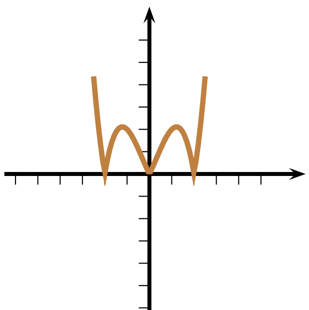


Figure D.160: D

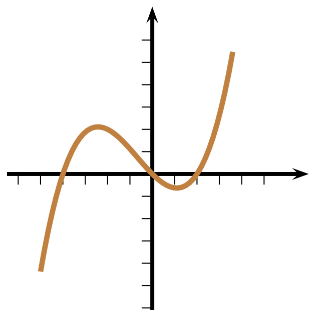


Figure D.161: E

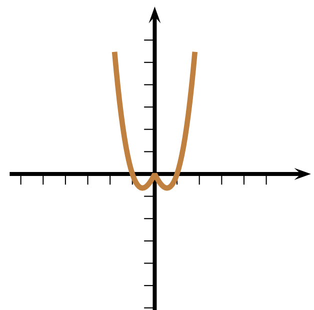


Figure D.162: F

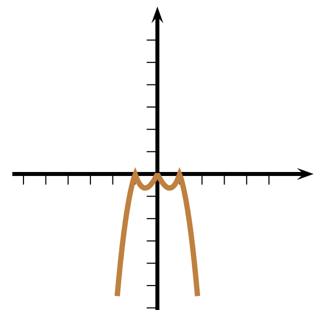


Figure D.163: G

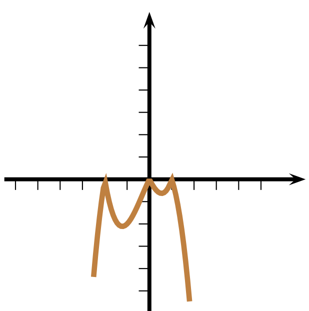


Figure D.164: H

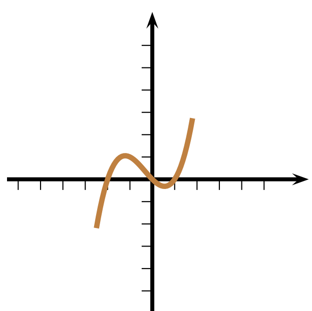


Figure D.165: I

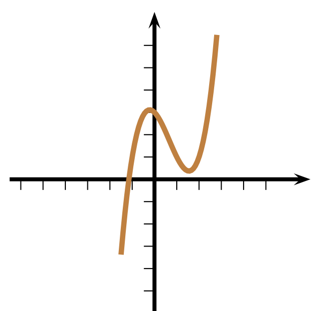


Figure D.166: J

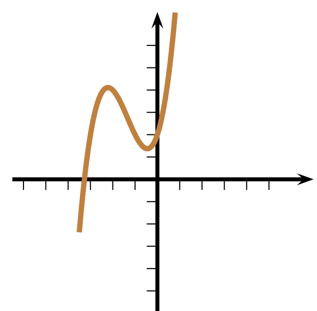


Figure D.167: K

α. $y = f(-x) =$ __________ β. $y = -f(-x) =$ __________ γ. $y = f(-|x|) =$ __________

δ. $y = f(x+1)+2 =$ __________ ε. $y = |f(-|x|)| =$ __________ ζ. $y = -|f(|x|)| =$ __________

η. $y = |f(-x)| =$ __________ θ. $y = |f(-|x|/2)| =$ __________ ι. $y = f(x/2) =$ __________

κ. $y = -|f(x)| =$ __________ λ. $y = \frac{1}{2}f(x) =$ __________ μ. $y = f(x-1)+1 =$ __________

You are to match the letters of figures D.168 to D.179 with the equations on 13 through 24 below. Some figures may not match with any equation, or viceversa. (0.5 mark each)

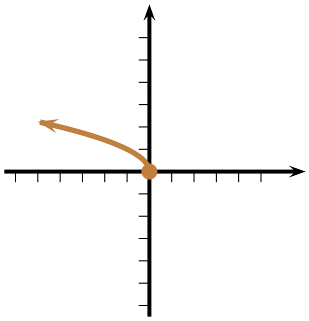


Figure D.168: A

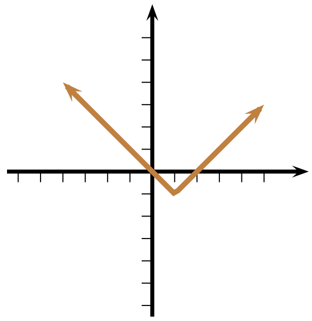


Figure D.169: B

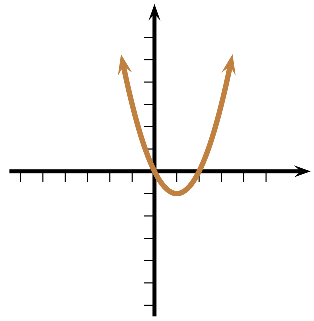


Figure D.170: C

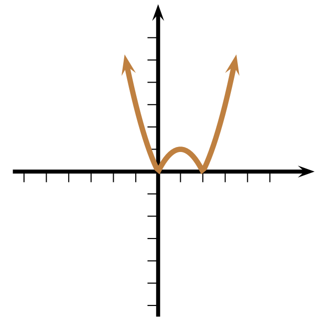


Figure D.171: D

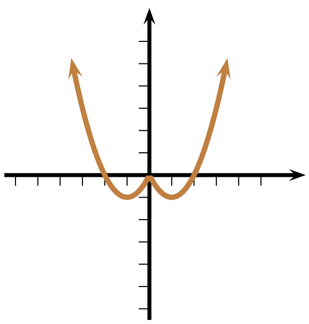


Figure D.172: E

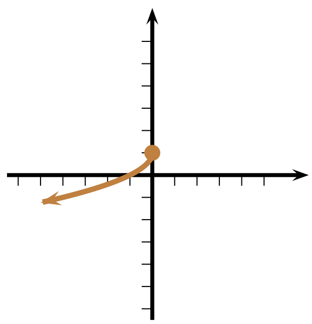


Figure D.173: F

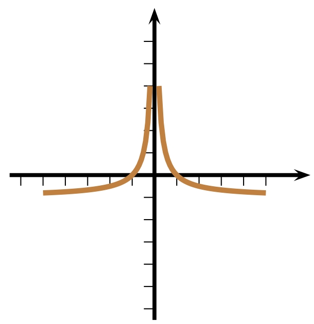


Figure D.174: G

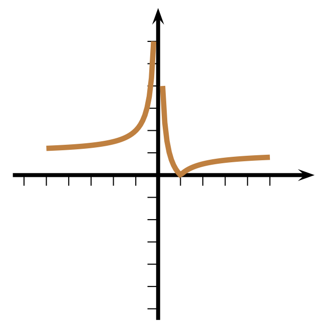


Figure D.175: H

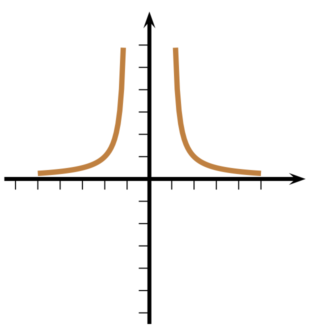


Figure D.176: I

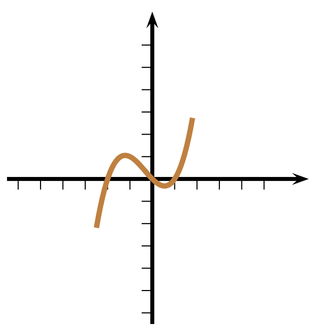


Figure D.177: J

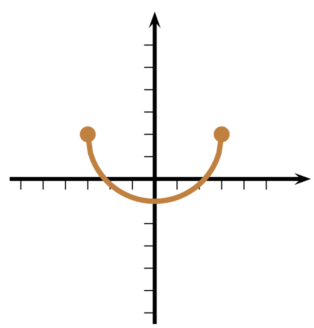


Figure D.178: K

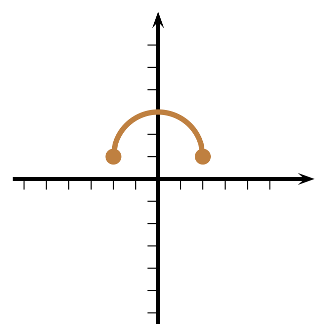


Figure D.179: L

13. $y = (x-1)^2 - 1 =$ ____________

14. $y = (|x|-1)^2 - 1 =$ ____________

15. $y = \sqrt{-x} =$ ____________

16. $y = |x-1| - 1 =$ ____________

17.$y = |(x-1)^2 - 1| =$ ____________

18. $y = 2 - \sqrt{9 - x^2} =$ ____________

19. $y = 1 + \sqrt{4 - x^2} =$ ____________

20.$y = |x^2 - 1| =$ ____________

21. $y = 1 - \sqrt{-x} =$ ____________

22. $y = \dfrac{1}{|x|} - 1 =$ ____________

23. $y = \left|\dfrac{1}{x} - 1\right| =$ ____________

24. $y = \dfrac{1}{|x|-1} =$ ____________

D.3 Essay Questions

1. Find the solution set to the inequality
$$\frac{(x-1)(x+2)}{(x-3)} \geq 0,$$
and write the answer in interval notation.

2. For the points $P(-1,2)$ and $Q(2,3)$, find:
   (a) the distance between P and Q,
   (b) the midpoint of the line segment joining P and Q,
   (c) if P and Q are the endpoints of a diameter of a circle, find the equation of the circle.

3. Show that if the graph of a curve has x-axis symmetry and y-axis symmetry then it must also have symmetry about the origin.

4. Consider the graph of the curve $y = f(x)$ in figure D.180. You may assume that the domain of f can be written in the form $[a;b[\cup]b;c]$, where a,b,c are integers, and that its range can be written in the form $[u;v]$, with u and v integers. Find a,b,c,u and v.

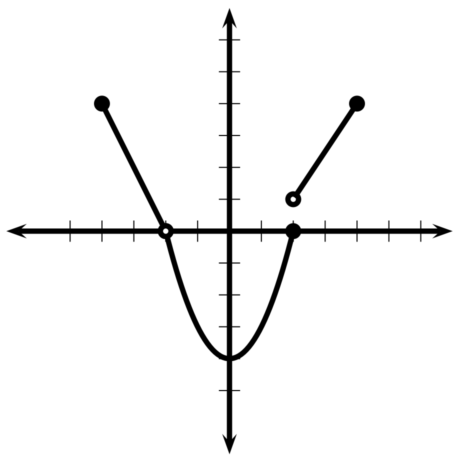


Figure D.180: Problem 4.

5. If the points $(1,3)$, $(-1,2)$, $(2,t)$ all lie on the same line, find the value of t.

6. An apartment building has 30 units. If all the units are inhabited, the rent for each unit is \$700 per unit. For every empty unit, management increases the rent of the remaining tenants by \$25. What will be the profit $P(x)$ that management gains when x units are empty? What is the maximum profit?

7. Draw a rough sketch of the graph of $y = x - \llbracket x \rrbracket$, where $\llbracket x \rrbracket$ is the the floor of x, that is, the greatest integer less than or equal to x.

8. Sketch the graphs of the curves in the order given. Explain, by which transformations (shifts, compressions, elongations, squaring, reflections, etc.) how one graph is obtained from the preceding one.
   (a) $y = x - 1$
   (b) $y = (x-1)^2$
   (c) $y = x^2 - 2x$
   (d) $y = |x^2 - 2x|$
   (e) $y = \dfrac{1}{|x^2 - 2x|}$

(f) $y = -\dfrac{1}{|x^2 - 2x|}$

(g) $y = \dfrac{1}{x^2 - 2|x|}$

9. The polynomial

$$p(x) = x^4 - 4x^3 + 4x^2 - 1$$

has a local maximum at $(1,0)$ and local minima at $(0,-1)$ and $(2,-1)$.

(a) Factor the polynomial completely and sketch its graph.

(b) Determine how many real zeros the polynomial $q(x) = p(x) + c$ has for each constant c.

10. The rational function q in figure D.181 has only two simple poles and satisfies $q(x) \to 1$ as $x \to \pm\infty$. You may assume that the poles and zeroes of q are located at integer points. Problems 10a to 10d refer to it.

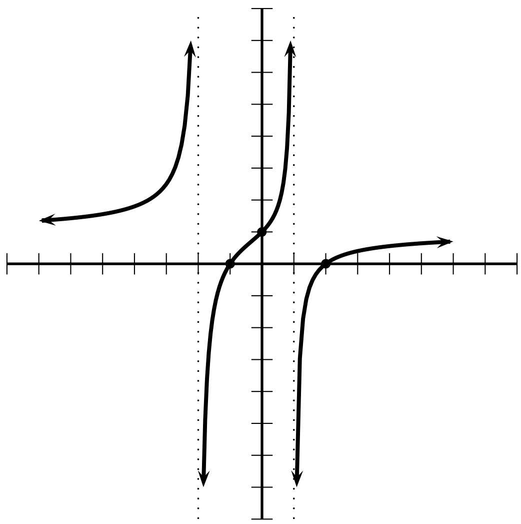


Figure D.181: Problems 10a to 10d.

(a) Find $q(0)$.

(b) Find $q(x)$ for arbitrary x.

(c) Find $q(-3)$.

(d) Find $\lim_{x\to-2+} q(x)$.

11. Find the solution to the absolute value inequality

$$|x^2 - 2x - 1| \leq 1,$$

and express your answer in interval notation.

12. Find all values of x for which the point $(x, x+1)$ is at distance 2 from $(-2,1)$.

13. Determine any intercepts with the axes and any symmetries of the curve

$$y^2 = |x^3 + 1|.$$

14. Let $f(x) = x^2$. Find

$$\frac{f(x+h) - f(x-h)}{h}.$$

15. **Situation:** Questions 15a to 15e refer to the straight line L_u given by the equation

$$L_u : \quad (u-2)y = (2u+4)x + 2u,$$

where u is a real parameter.

(a) For which value of u is L_u a horizontal line?

(b) For which value of u is L_u a vertical line?

(c) For which value of u is L_u parallel to the line $y = -2x+1$?

(d) For which value of u is L_u perpendicular to the line $y = -2x+1$?

(e) Is there a point which is on every line L_u regardless the value of u? If so, find it. If not, prove that there is no such point.

16. The polynomial p in figure D.182 has degree 3. You may assume that all its roots are integers. Problems 16a to 16b refer to it.

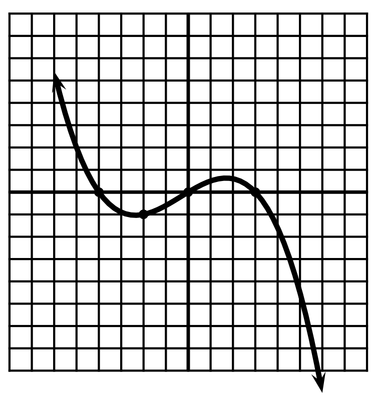


Figure D.182: Problems 16a to 16b.

(a) Find $p(-2)$, assuming it is an integer.

(b) Find a formula for $p(x)$.

17. A rectangular box with a square base of length x and height h is to have a volume of 20 ft^3. The cost of the material for the top and bottom of the box is 20 cents per square foot. Also, the cost of the material for the sides is 8 cents per square foot. Express the cost of the box in terms of

(a) the variables x and h;

(b) the variable x only; and

(c) the variable h only.

18. Sketch the graph of the curve $y = \sqrt{\dfrac{1-x}{x+1}}$ and label the axis intercepts and asymptotes.

19. Find all the rational roots of $x^5 + 4x^4 + 3x^3 - x^2 - 4x - 3 = 0$.

20. Given $f(x) = \dfrac{1}{x+1}$, graph

(a) $y = |f(x)|$,

(b) $y = f(|x|)$,

(c) $y = |f(|x|)$,

(d) $y = f(-|x|)$.

21. Graph $y = (x-1)^{2/3} + 2$ noting any intercepts with the axes.

Problems 22 through 29 refer to the curve with equation $y = |x+2| + |x-3|$**.**

22. Write the equation $y = |x+2| + |x-3|$ without absolute values if $x \leq -2$.

23. Write the equation $y = |x+2| + |x-3|$ without absolute values if $-2 \leq x \leq 3$.

24. Write the equation $y = |x+2| + |x-3|$ without absolute values if $x \geq 3$.

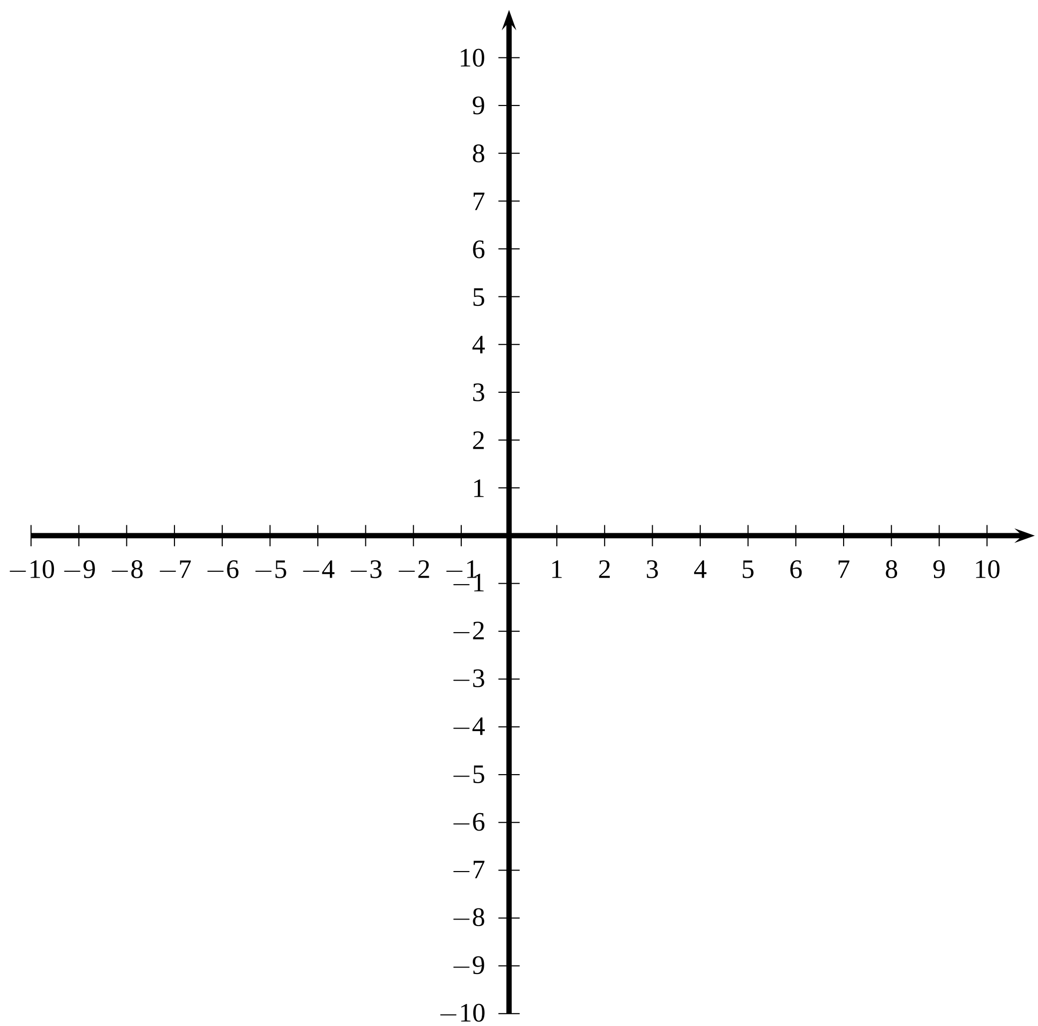
25. Solve the equation $|x+2| + |x-3| = 7$.

26. Solve the equation $|x+2| + |x-3| = 4$.

27. Graph the curve $y = |x+2| + |x-3|$ on the axes below. Use a ruler or the edge of your ID card to draw the straight lines.

28. Graph the curve $y = 4$ on the axes below.

29. Graph the curve $y = 7$ on the axes above.



Questions 30 through 32 refer to the circle $\mathscr{C}$ having centre at $O(1,2)$ and passing through the point $A(5,5)$, as shewn in figure D.183 below.

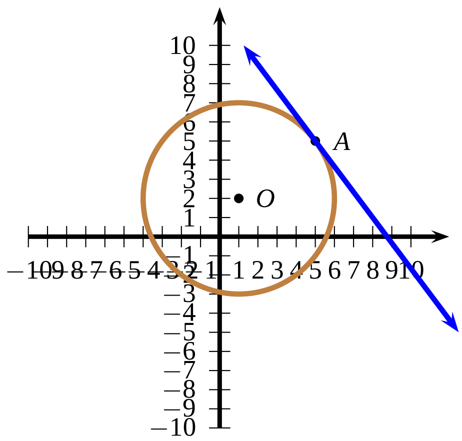


Figure D.183: Problems 30 through 32 .

30. Find the equation of the circle $\mathscr{C}$.
31. If the point $(2,a)$ is on the circle $\mathscr{C}$, find all the possible values of a.
32. Find the equation of the line L that is tangent to the circle $\mathscr{C}$ at A. (Hint: A tangent to a circle at a point is perpendicular to the radius passing through that point.)

Problems 34 through 39 refer to the graph of a function f is given in figure D.184.

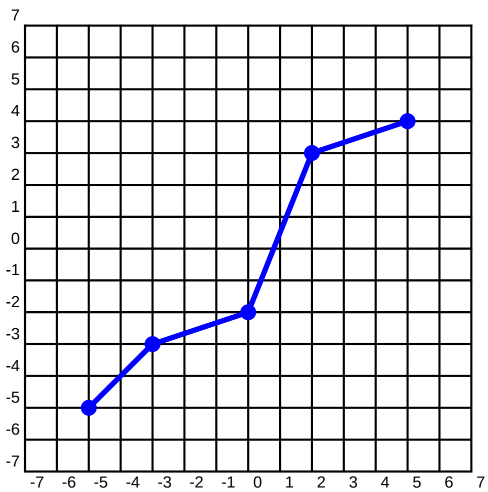


Figure D.184: Problems 34 through 39.

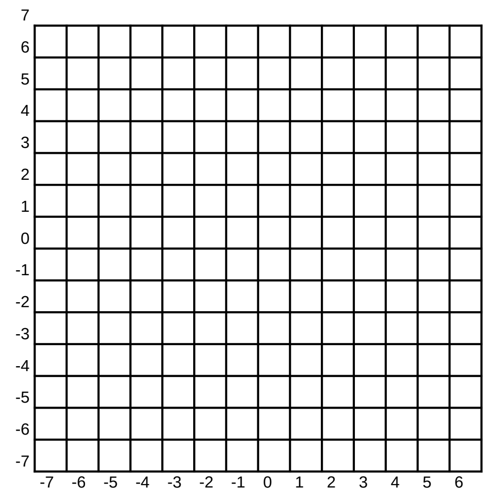


Figure D.185: Problems 34 through 39.

33. Give a brief explanation as to why f is invertible.
34. Determine $\textbf{Dom}(f)$.
35. Determine $\textbf{Im}(f)$.

36. Draw the graph of f^{-1} in figure D.185.
37. Determine $f(-5)$.
38. Determine $f^{-1}(3)$.
39. Determine $f^{-1}(4)$.

Figure D.186 has the graph of a curve $y = f(x)$. Draw each of the required curves *very carefully*.

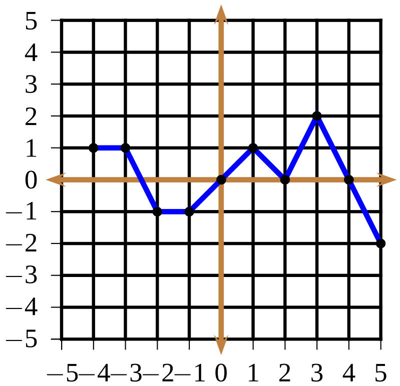


Figure D.186: $y = f(x)$.

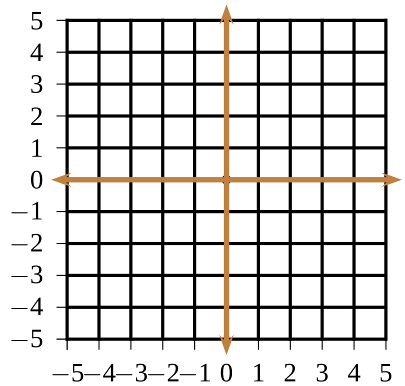


Figure D.187: $y = f(x) + 1$.

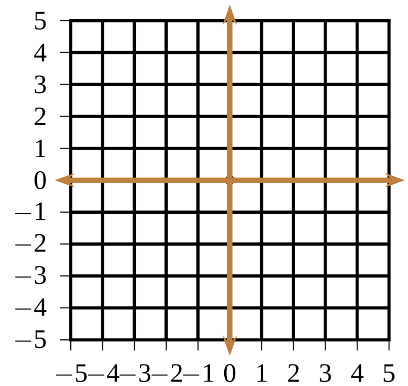


Figure D.188: $y = |f(x) + 1|$.

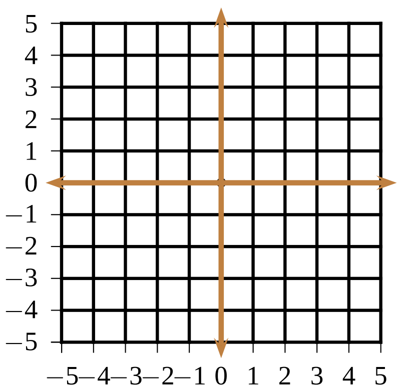


Figure D.189: $y = f(-|x|)$.

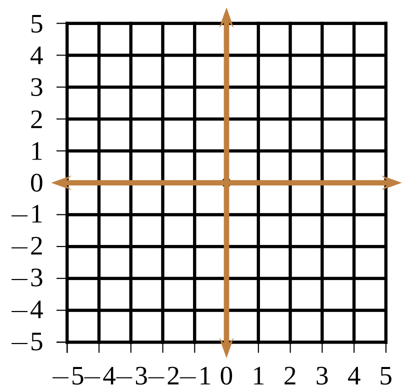


Figure D.190: $y = |f(-|x|)|$.

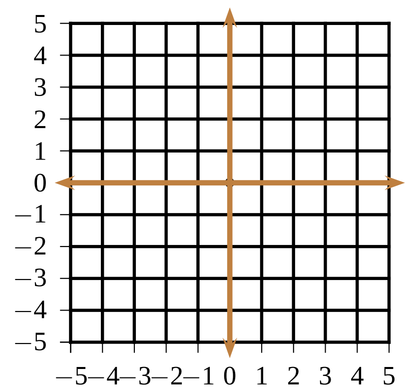


Figure D.191: $y = -f(-x)$.

40. Figure D.198 has the graph of a curve $y = f(x)$, which is composed of lines and a pair of semicircles. Draw each of the required curves *very carefully*. Use a ruler or the edge of your id card in order to draw the lines. Shapes with incorrect coordinate points will not be given credit.

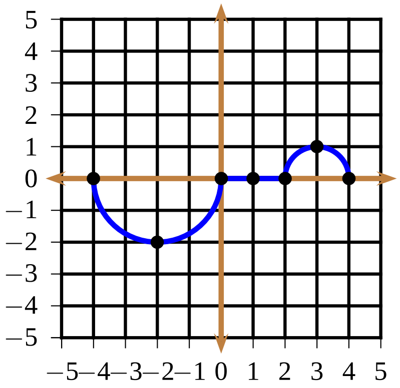


Figure D.192: $y = f(x)$.

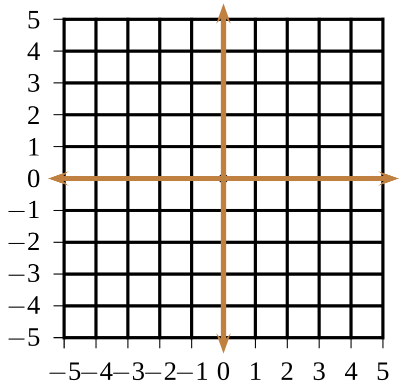


Figure D.193: $y = f(-x)$.

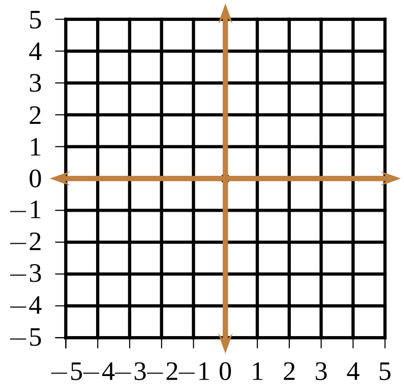


Figure D.194: $y = -f(x)$.

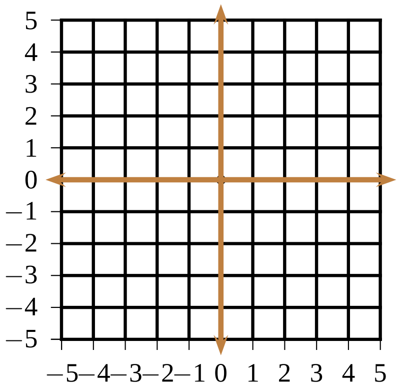


Figure D.195: $y = |f(x)|$.

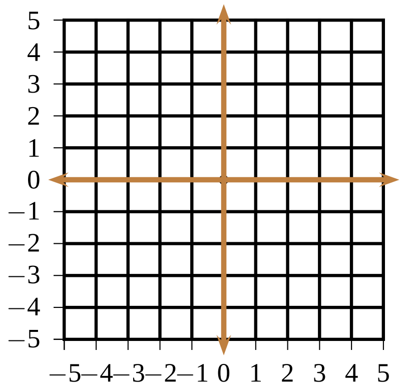


Figure D.196: $y = f(-|x|)$.

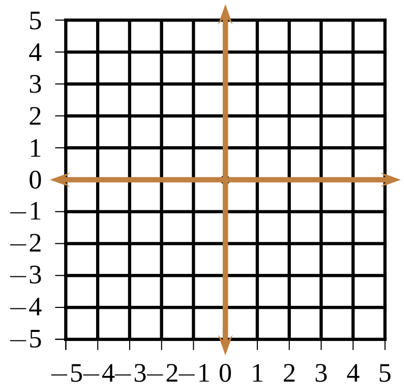


Figure D.197: $y = f(|x|)$.

41. Use the following set of axes to draw the following curves in succession. Note all intercepts.

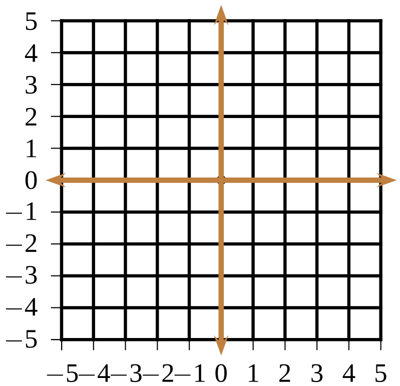


Figure D.198: $y = x - 2$.

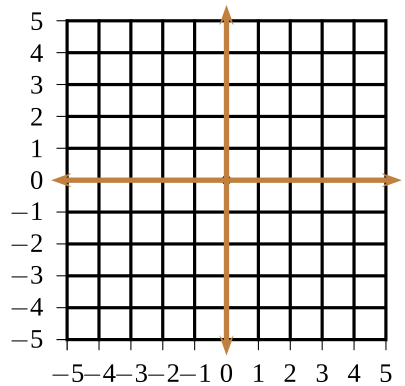


Figure D.199: $y = |x - 2|$.

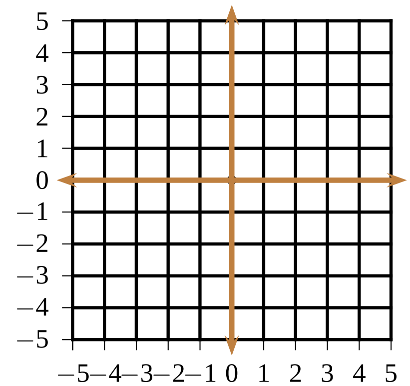


Figure D.200: $y = |x| - 2$.

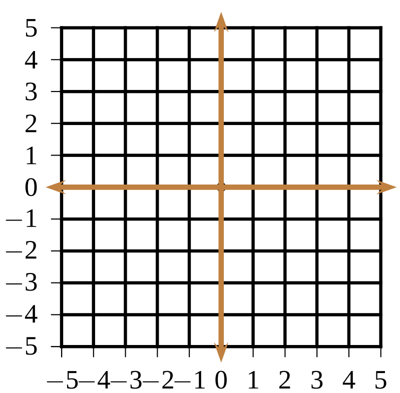


Figure D.201: $y = ||x| - 2|$.

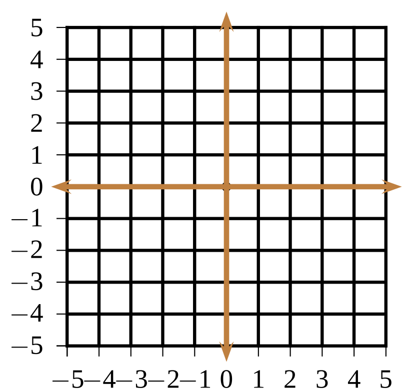


Figure D.202: $y = |-|x| - 2|$.

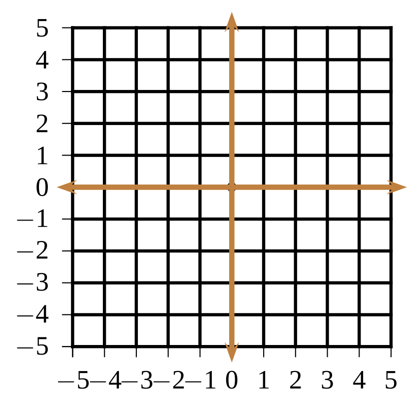


Figure D.203: $|y| = x - 2$.

Situation: $\triangle ABC$ is right-angled at A, and $AB = 2$ and $\tan\angle B = \frac{1}{2}$. Problems 42 through 45 refer to this situation.

42. Find AC.

43. Find BC.

44. Find $\sin\angle B$.

45. Find $\tan\angle C$.

46. Using the standard labels for a $\triangle ABC$, prove that $\dfrac{a-b}{a+b} = \dfrac{\sin A - \sin B}{\sin A + \sin B}$.

47. A triangle has sides measuring $2, 3, 4$. Find the cosine of the angle opposite the side measuring 3.

48. Find the area of a triangle whose sides measure $2, 3, 4$. Find the radius of its circumcircle.

49. If in a $\triangle ABC$, $a = 5$, $b = 4$, and $\cos(A - B) = \dfrac{31}{32}$, prove that $\cos C = \dfrac{1}{8}$ and that $c = 6$.

50. A triangle with vertices A, B, C on a circle of radius R, has the side opposite to vertex A of length 12, and the angle at $A = \frac{\pi}{4}$. Find diameter of the circle.

51. $\triangle ABC$ has sides of length a,b,c, and circumradius $R=4$. Given that the triangle has area 5, find the product abc.

52. Find, approximately, the area of a triangle having two sides measuring 1 and 2 respectively, and angle between these sides measuring 35°. What is the measure of the third side?

53. Find the area and the perimeter of a regular octagon inscribed in a circle of radius 2.

54. Two buildings on opposite sides of a street are 45 m apart. From the top of the taller building, which is 218 m high, the angle of depression to the top of the shorter building is 13.75°. Find the height of the shorter building.

55. A ship travels for 3 hours at 18 mph in a direction N28°E. From its current direction, the ship then turns through an angle of 95° to the right and continues traveling at 18 mph. How long will it take before the ship reaches a point directly east of its starting point?

56. Let $\tan x+\cot x=a$. Find $\tan^3 x+\cot^3 x$ as a polynomial in a.

57. If $\cos\dfrac{\pi}{7}=a$, find the exact value of $\cos\dfrac{\pi}{14}$ and $\cos\dfrac{2\pi}{7}$ in terms of a.

58. Given that $\csc x=-4$, and $\mathscr{C}$ x lies in quadrant III, find the remaining trigonometric functions.

59. Graph the curve $y=2-\cos\dfrac{x}{2}$.

60. Graph the curve $y=\left|2-\cos\dfrac{x}{2}\right|$.

61. Find the smallest positive solution, if any, to the equation $3^{\cos 3x}=2$. Approximate this solution to two decimal places.

62. Find all the solutions lying in $[0;2\pi]$ of the following equations:

   (a) $2\sin^2 x+\cos x-1=0$
   (b) $\sin 2x=\cos x$
   (c) $\sin 2x=\sin x$
   (d) $\tan x+\cot x=2\csc 2x$

63. Find the exact value of $\sin\dfrac{88\pi}{3}$.

64. Find the exact value of $\tan\left(\arcsin\dfrac{1}{3}\right)$.

65. Is $\sin(\arcsin 30)$ a real number?

66. Find the exact value of $\arcsin(\sin 30)$.

67. Find the exact value of $\arcsin(\cos 30)$.

68. If x and y are acute angles and $\sin\dfrac{x}{2}=\dfrac{1}{3}$ and $\cos y=\dfrac{3}{4}$, find the exact value of $\tan(x-y)$.

69. Find the exact value of the product
$$P=\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}.$$

70. How many digits does $5^{2000}3^{1000}$ have?

71. What is $5^{2000}3^{1000}$ approximately?

72. Let $a>1, x>1, y>1$. If $\log_a x^3=N$ and $\log_{a^{1/3}} y^4=M$, find $\log_{a^2} xy$ in terms of N and M. Also, find $\log_x y$.

73. Graph $y=3^{-x}-2$.

74. Graph $y=3^{-|x|}-2$.

75. Graph $y=|3^{-x}-2|$.

76. Graph $y = \ln(x+1)$.
77. Graph $y = \ln(|x|+1)$.
78. Graph $y = |\ln(x+1)|$.
79. Graph $y = |\ln|(x+1)||$.
80. Solve the equation $3^x + \dfrac{1}{3^x} = 12$.
81. The expression
$$(\log_2 3)\cdot(\log_3 4)\cdot(\log_4 5)\cdots(\log_{511} 512)$$
is an integer. Find it.
82. The expression
$$\log(\tan 1^\circ) + \log(\tan 2^\circ) + \log(\tan 3^\circ) + \cdots + \log(\tan 89^\circ)$$
is an integer. Find it.
83. Prove that the equation
$$\cos\left(\left(\frac{3}{2}\right)^x - 1\right) = \frac{1}{2},$$
has only 4 solutions lying in the interval $[0;2\pi]$.
84. Prove that the equation
$$\cos(\log_3 x - 2) = \frac{1}{2},$$
has only 2 solutions lying in the interval $[0;2\pi]$.

E Maple

The purpose of these labs is to familiarise you with the basic operations and commands of Maple. The commands used here can run on any version of Maple (at least V through X).

E.1 Basic Arithmetic Commands

Maple uses the basic commands found in most calculators: $+$ for addition, $-$ for subtraction, $*$ for multiplication, $/$ for division, and $\wedge$ for exponentiation. Maple also has other useful commands like `expand` and `simplify`. Be careful with capitalisation, as Maple distinguishes between capital and lower case letters. For example, to expand the algebraic expression $(\sqrt{8}-2^{1/2})^2$, type the following, pressing $\boxed{\text{ENTER}}$ after the semicolon:

F Some Answers and Solutions

Answers

1.1.1 This is the set $\{-9,-6,-3,3,6,9\}$.

1.1.2 We have
$$x^2-x=6 \implies x^2-x-6=0 \implies (x-3)(x+2)=0 \implies x\in\{-2,3\}.$$
Since $-2\not\in\mathbb{N}$, we deduce that
$$\{x\in\mathbb{N}: x^2-x=6,\}=\{3\}.$$

1.1.3 We have
$$2<\frac{x}{6}<3 \implies 12<x<18 \implies x\in\{13,14,15,16,17\}.$$

1.1.4 $A\cup B=\{a,b,c,d,e,f,i,o,u\}$, $A\cap B=\{a,e\}$, $A\setminus B=\{b,c,d,f\}$, $B\setminus A=\{i,o,u\}$

1.1.5 (i) $\{2\}$, (ii) $\{-2,2\}$, (iii) $\varnothing$, (iv) $]-4;4[$, (v) $\{-3,-2,-1,0,1,2,3\}$, (vi) $]-1;1[$, (vii) $\{0\}$, (viii) $\varnothing$

1.1.6 $\{-32,-33,-34,\ -35,-36,\ -37,-38,-39,\ -40,-41,-42,-43,-44\}$

1.1.7 Observe that applying k times the second rule, $n+5k$ is in S. Similarly, $3^k\cdot 2$ is in S by applying k times the third rule. Since 2 is in S, $2+5k$ is in S, that is, numbers that leave remainder 2 upon division by 5 are in S. This means that
$$\{2,7,12,\cdots 2002,2007\}\subseteq S.$$
Since $3\cdot 2=6$ is in S, then the numbers $6+5k=1+5(k+1)$ are in S, that is, numbers 6 or higher that leave remainder 1 upon division by 5. Thus the numbers
$$\{6,11,16,\cdots 2001,2006\}\subseteq S.$$
Since $3\cdot 6=18$ is in S, then the numbers $18+5k=3+5(k+3)$ are in S, that is, numbers 18 or higher that leave remainder 3 upon division by 5. Thus the numbers
$$\{18,23,28,\cdots 2003,2008\}\subseteq S.$$
Since $3\cdot 18=54$ is in S, then the numbers $54+5k=4+5(k+10)$ are in S, that is, numbers 54 or higher that leave remainder 4 upon division by 5. Thus the numbers
$$\{54,59,64,\cdots 2004\}\subseteq S.$$
Now, we claim that there are no multiples of 5 in S. For by combining the rules every number in S has the form $3^a\cdot 2+5b$, with $a\geq 0$, $b\geq 0$ integers. Since $3^a\cdot 2$ is never a multiple of 5, this establishes the claim. Hence the largest element of
$$\{1,2,3,\ldots,2008\}$$
not in S is 2005.

1.1.9 $]-1;5[$, $]-5;+\infty[$, $]-5;-1]$, $[5;+\infty[$

1.1.10 $\varnothing$, $]-5;3[\cup[4;+\infty[$, $]-5;3[$, $[4;+\infty[$

1.1.11 $[-0.5;-2+\sqrt{3}[$, $\left[-1;\sqrt{2}-1\right]$, $[-1;-0.5[$, $\left[-2+\sqrt{3};\sqrt{2}-1\right]$

1.1.13 Hint: Consider the $N+1$ numbers $tx-\lfloor tx\rfloor, t=0,1,2,\ldots,N$.

1.2.1 If $x=0.123123123\ldots$ then $1000x=0.123123123\ldots$ giving $1000x-x=123$, since the tails cancel out. This results in $x=\dfrac{123}{999}=\dfrac{41}{333}$.

1.2.2 If $\sqrt{8}=2\sqrt{2}$ were rational, then there would exist strictly positive natural numbers a,b such that $2\sqrt{2}=\dfrac{a}{b}$, which entails that $\sqrt{2}=\dfrac{a}{2b}$, a rational number, a contradiction.

1.2.3 If $\sqrt{2}+\sqrt{3}$ were rational, then there would exist strictly positive natural numbers a, b such that $\sqrt{2}+\sqrt{3}=\frac{a}{b}$, which entails that

$$(\sqrt{2}+\sqrt{3})^2=\frac{a^2}{b^2} \implies 2+2\sqrt{6}+3=\frac{a^2}{b^2} \implies \sqrt{6}=\frac{a^2}{2b^2}-\frac{5}{2}.$$

The dextral side of the last equality is a rational number, but the sinistral side is presumed irrational, a contradiction.

1.2.4 Yes! There multiple ways of doing this. An idea is to take the first decimal digits of $\sqrt{2}$, remove them, and supplant them with the given string. For example, for 12345 we proceed as follows:

$$\sqrt{2}\approx 1.414213562\ldots \implies \frac{\sqrt{2}}{10^6}\approx 0.000001414213562\ldots.$$

Then the number

$$\frac{\sqrt{2}}{10^6}+0.12345$$

is an irrational number whose first five digits after the decimal point are 12345.

Another idea is to form the number

$$0.1234501234500123450000123450000000012345\ldots$$

where one puts 2^k zeroes between appearances of the string 12345.

1.2.5 There are infinitely many answers. Since $\sqrt{2}<1.5$ and $1.7<\sqrt{3}$, we may take, say, 1.6. Of course, using the mentioned inequalities we may take also 1.61, 1.601, 1.52, etc.

1.2.6 There are infinitely many answers. One may take the average, $\frac{\sqrt{2}+\sqrt{3}}{2}$.

1.2.7 Since $\frac{1}{10}=0.1$ and $0.111<\frac{1}{9}=$, we may take, say $0.110100100001000000001\ldots$, where there are $2^k, k=1,2,\ldots$ 0's between consecutive 1's. Another approach can be taking $\sqrt{2}-1.314$, since $\sqrt{2}-1.314<0.1003$.

1.3.1 We have,

$$\begin{aligned}\left(\frac{2}{x}+\frac{x}{2}\right)^2-\left(\frac{2}{x}-\frac{x}{2}\right)^2 &= \left(\frac{4}{x^2}+2+\frac{x^2}{4}\right)-\left(\frac{4}{x^2}-2+\frac{x^2}{4}\right)\\ &= 4.\end{aligned}$$

1.3.2 We have

$$1=(x+y)^2=x^2+y^2+2xy=x^2+y^2-4 \implies x^2+y^2=5.$$

Hence,

$$(x-y)^2=x^2+y^2-2xy=5-2(-2)=9 \implies x-y=\pm 3.$$

In the first case,

$$x+y=1, x-y=3 \implies x=2, \quad y=-1.$$

In the second case,

$$x+y=1, x-y=-3 \implies x=-1, \quad y=2.$$

1.3.3 We have

$$7=x^3+y^3=(x+y)(x^2-xy+y^2)=x^2-xy+y^2.$$

Also,

$$1=(x+y)^2=x^2+y^2+2xy.$$

This gives

$$6=(x^2-xy+y^2)-(x^2+2xy+y^2)=-3xy \implies -2=xy.$$

Hence we have the system

$$x+y=1, \quad xy=-2,$$

which was already solved in problem 1.3.2.

1.3.4 We have

$$\begin{aligned}1^2-2^2+3^2-4^2+\cdots+99^2-100^2 &= (1-2)(1+2)+(3-4)(3+4)+\cdots+(99-100)(99+100)\\ &= -(1+2+3+4+\cdots+99+100)\end{aligned}$$

To compute the sum of the arithmetic progression $1+2+3+4+\cdots+99+100$, use Gauß 's trick: if $S=1+2+3+4+\cdots+99+100$, then $S=100+99+\cdots+2+1$. Hence

$$2S=(1+100)+(2+99)+\cdots+(99+2)+(100+1)=101\cdot 100 \implies S=5050.$$

This means that

$$1^2-2^2+3^2-4^2+\cdots+99^2-100^2=-5050.$$

1.3.5 Since $n^3-8=(n-2)(n^2+2n+4)$, for it to be a prime one needs either $n-2=1 \implies n=3$ or $n^2+2n+4=1$, but this last equation does not have integral solutions. Hence $3^3-8=19$ is the only such prime.

1.3.6 Put $x=1234567890$. Then

$$1234567890^2-1234567889\cdot 1234567891=x^2-(x-1)(x+1)=x^2-(x^2-1)=1.$$

1.3.7 If the numbers are x,y then $x+y=3$ and $xy=9$. This gives

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=\frac{3}{9}=\frac{1}{3}.$$

1.3.8 We have

$$\begin{aligned}1,\ 000,\ 002,\ 000,\ 001 &= 10^{12}+2\cdot 10^6+1\\ &= (10^6+1)^2\\ &= ((10^2)^3+1)^2\\ &= (10^2+1)^2((10^2)^2-10^2+1)^2\\ &= 101^2 9901^2,\end{aligned}$$

whence the prime sought is 9901.

1.3.10 One can expand the dextral side and obtain the sinistral side.

$$x^3+y^3=(x+y)^3-3xy(x+y)$$

twice:

$$\begin{aligned}a^3+b^3+c^3-3abc &= (a+b)^3+c^3-3ab(a+b)-3abc\\ &= (a+b+c)^3-3(a+b)c(a+b+c)-3ab(a+b+c)\\ &= (a+b+c)((a+b+c)^2-3ac-3bc-3ab)\\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca).\end{aligned}$$

1.3.11 From problem 1.3.9,

$$36=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)=a^2+b^2+c^2+4 \implies a^2+b^2+c^2=32.$$

From problem 1.3.10,

$$abc=\frac{a^3+b^3+c^3-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{3}=\frac{6-(-6)(32-2)}{3}=62.$$

1.3.12 Put $x = 1\,000\,000 = 10^6$. Then

$$x(x+1)(x+2)(x+3) = x(x+3)(x+1)(x+2) = (x^2+3x)(x^2+3x+2).$$

Again, put $y = x^2+3x$. Then

$$x(x+1)(x+2)(x+3)+1 = (x^2+3x)(x^2+3x+2)+1 = y(y+2)+1 = (y+1)^2.$$

All this gives,

$$\begin{aligned}\sqrt{x(x+1)(x+2)(x+3)+1} &= y+1\\ &= x^2+3x+1\\ &= 10^{12}+3\cdot 10^6+1\\ &= 1\,000\,003\,000\,001.\end{aligned}$$

1.3.13 We have,

$$\sqrt{5+2\sqrt{6}} = \sqrt{2+2\sqrt{6}+3} = \sqrt{(\sqrt{2}+\sqrt{3})^2} = \sqrt{2}+\sqrt{3}$$

1.3.14 Transposing,

$$a^2-ab+b^2-bc+c^2-dc+d^2-da = 0,$$

or

$$\frac{a^2}{2}-ab+\frac{b^2}{2}+\frac{b^2}{2}-bc+\frac{c^2}{2}+\frac{c^2}{2}-dc+\frac{d^2}{2}+\frac{d^2}{2}-da+\frac{a^2}{2} = 0.$$

Factoring,

$$\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-d)^2+\frac{1}{2}(d-a)^2 = 0.$$

As the sum of positive quantities is zero only when the quantities themselves are zero, we obtain $a = b, b = c, c = d, d = a$, which proves the assertion.

1.3.15 We have

$$\begin{aligned}(x+y)^2 = (x-1)(y+1) &\implies (x-1+y+1)^2 = (x-1)(y+1)\\ &\implies (x-1)^2+2(x-1)(y+1)+(y+1)^2 = (x-1)(y+1)\\ &\implies (x-1)^2+(x-1)(y+1)+(y+1)^2 = 0\\ &\implies \left(x-1+\frac{y+1}{2}\right)^2+\frac{3(y+1)^2}{4} = 0.\end{aligned}$$

This last is a sum of squares, which can only be zero if

$$x-1+\frac{y+1}{2} = 0, \qquad y+1 = 0 \implies x = 1, y = -1.$$

Thus $(x,y) = (1,-1)$ is the only solution.

1.3.16 Observe that

$$\begin{aligned}\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} &= \frac{a^2+b^2}{-c}+\frac{b^2+c^2}{-a}+\frac{c^2+a^2}{-b}\\ &= -a^2\left(\frac{1}{b}+\frac{1}{c}\right)-b^2\left(\frac{1}{c}+\frac{1}{a}\right)-c^2\left(\frac{1}{a}+\frac{1}{b}\right)\\ &= -a^2\left(\frac{c+b}{bc}\right)-b^2\left(\frac{a+c}{ca}\right)-c^2\left(\frac{b+a}{ab}\right)\\ &= \frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab},\end{aligned}$$

as was to be shewn.

1.3.17 Put $S = 1+a+a^2+\cdots+a^{n-1}$. Then $aS = a+a^2+\cdots+a^{n-1}+a^n$. Thus

$$S-aS = (1+a+a^2+\cdots+a^{n-1})-(a+a^2+\cdots+a^{n-1}+a^n) = 1-a^n,$$

and from $(1-a)S = S-aS = 1-a^n$ we obtain the result.

By making the substitution $a = \frac{y}{x}$ in the preceding identity, we see that

$$1+\frac{y}{x}+\left(\frac{y}{x}\right)^2+\cdots+\left(\frac{y}{x}\right)^{n-1} = \frac{1-\left(\frac{y}{x}\right)^n}{1-\frac{y}{x}}$$

we obtain

$$\left(1-\frac{y}{x}\right)\left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2+\cdots+\left(\frac{y}{x}\right)^{n-1}\right) = 1-\left(\frac{y}{x}\right)^n,$$

or equivalently,

$$\left(1-\frac{y}{x}\right)\left(1+\frac{y}{x}+\frac{y^2}{x^2}+\cdots+\frac{y^{n-1}}{x^{n-1}}\right) = 1-\frac{y^n}{x^n}.$$

Multiplying by x^n both sides,

$$x\left(1-\frac{y}{x}\right)x^{n-1}\left(1+\frac{y}{x}+\frac{y^2}{x^2}+\cdots+\frac{y^{n-1}}{x^{n-1}}\right) = x^n\left(1-\frac{y^n}{x^n}\right),$$

which is

$$x^n-y^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}),$$

yielding the result.

1.3.18 Observe that

$$(ac+bd)^2+(ad-bc)^2 = a^2c^2+b^2d^2+a^2d^2+b^2c^2 = (a^2+b^2)(c^2+d^2).$$

1.4.1

1. Observe that $x^2-x-6 = (x-3)(x+2)$. Hence, in a neighbourhood of $x=-2$ and $x=3$, we have:

$x\in$	$]-\infty[-2$	$]-2;3[$	$]3;+\infty[$
$x+2$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$(x+2)(x-3)$	$+$	$-$	$+$

2. From the diagram we deduce that the desired set is $[-2;3]$.
3. From the diagram we deduce that the desired set is $]-\infty;-2[\cup[3;+\infty[$.

1.4.2 $\{x\in\mathbb{R} : x\in\,]-\infty;-3]\cup]-2;2]\cup]3;+\infty[\}$.

1.4.3 As the equation $x^2-x-4=0$ does not have rational roots, we complete squares to find its roots:

$$x^2-x-4 = x^2-x+\frac{1}{4}-\frac{1}{4}-4 = \left(x-\frac{1}{2}\right)^2-\frac{17}{4} = \left(x-\frac{1}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{1}{2}+\frac{\sqrt{17}}{2}\right).$$

Therefore

$$x^2-x-4\geq 0 \iff \left(x-\frac{1}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{1}{2}+\frac{\sqrt{17}}{2}\right)\geq 0.$$

We may now form a sign diagram, puncturing the real line at $x=\frac{1}{2}-\frac{\sqrt{17}}{2}$ and at $x=\frac{1}{2}+\frac{\sqrt{17}}{2}$:

$x\in$	$\left]-\infty;\frac{1}{2}-\frac{\sqrt{17}}{2}\right[$	$\left]\frac{1}{2}-\frac{\sqrt{17}}{2};\frac{1}{2}+\frac{\sqrt{17}}{2}\right[$	$\left]\frac{1}{2}+\frac{\sqrt{17}}{2};+\infty\right[$
$\left(x-\frac{1}{2}-\frac{\sqrt{17}}{2}\right)$	$-$	$+$	$+$
$\left(x-\frac{1}{2}+\frac{\sqrt{17}}{2}\right)$	$-$	$-$	$+$
$\left(x-\frac{1}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{1}{2}+\frac{\sqrt{17}}{2}\right)$	$+$	$-$	$+$

We deduce that

$$\left\{x\in\mathbb{R} : x^2-x-4\geq 0\right\} = \left]-\infty;\frac{1}{2}-\frac{\sqrt{17}}{2}\right[\cup\left]\frac{1}{2}+\frac{\sqrt{17}}{2};+\infty\right[.$$

1.4.4 $[-2;-1]$

1.4.5 $[3;+\infty[$

1.4.6 $\left]\frac{12}{25};\frac{1}{2}\right]$

1.4.7 $\left\{\frac{5}{2}\right\}$

1.4.8 From the identity $x^2-y^2=(x-y)(x+y)$ and using the fact that $\sqrt{n}<\sqrt{n+1}$, we obtain

$$n+1-n=1 \implies (\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})=1 \implies \sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}>\frac{1}{2\sqrt{n+1}}.$$

Hence,

$$\frac{1}{2\sqrt{n+1}}<\frac{1}{10} \implies 5<\sqrt{n+1} \implies 25<n+1 \implies n>24.$$

Since $5.1^2=26.01>26$, we have $\sqrt{26}<5.1$. Hence,

$$\sqrt{26}-\sqrt{25}<5.1-5=\frac{1}{10},$$

and so $n=25$ fulfills the inequality.

1.4.9 The quadratic equation will not have any real solutions as long as its determinant be strictly negative:

$$t^2-(t-1)(t)<0 \implies t\in\,]-\infty;0[\,.$$

Hence $A_t=\varnothing \iff t\in\,]-\infty;0[$.
The set will have exactly one element either when $t-1=0$, which means that the equation reduces to a linear one, or if the quadratic equation has a repeated root, which occurs when its discriminant vanishes:

$$t^2-(t-1)(t)=0 \implies t=0.$$

Thus the set has exactly one element when $t=1$ and when $t=0$, and it is seen that

$$A_1=\{x\in\mathbb{R}x+\frac{1}{4}=0\}=\{-\frac{1}{4}\}, \qquad A_0=\{x\in\mathbb{R}-x^2=0\}=\{0\}.$$

The quadratic equation will have exactly two real solutions when its discriminant is strictly positive:

$$t-1\neq 0, t^2-(t-1)(t)>0 \implies t\in\,]0;1[\cup]1;+\infty[\,.$$

In this case,

$$A_t=\left\{x\in\mathbb{R}:(t-1)x^2+tx+\frac{t}{4}=0\right\}=\left\{\frac{4\sqrt{t}-4t}{8t-8},\frac{-4\sqrt{t}-4t}{8t-8}\right\}.$$

1.4.10 $\{-1,0,1,2,3,4,5,6,7,8,9\}$

1.4.11 The inequality is obtained at once from

$$2x^3-6x^2+\frac{11}{2}x+1>=x\left(\left(\sqrt{2}x-\frac{3}{\sqrt{2}}\right)^2+1\right)+1.$$

1.4.12 Either $x\in\,]-\infty;0]$, or $x\in\,]0;1]$, or $x\in\,]1;+\infty[$. In the first case the inequality is obvious, since for $x\leq 0$

$$x^8\geq 0,-x^5\geq 0,x^2\geq 0,-x\geq 0 \implies x^8-x^5+x^2-x+1>0.$$

In the second case we regroup

$$x^8-x^5+x^2-x+1=x^8+x^2(1-x^3)+(1-x)>0.$$

In the third case we have

$$x^8-x^5+x^2-x+1=x^5(x^3-1)+x(x-1)+1>0$$

1.4.13 25

1.5.1 $\sqrt{3}-\sqrt{\sqrt{15}-2}$

1.5.2 For $x > \frac{1}{2}$, we have $|1-2x| = 2x-1$. Thus $|x-|1-2x|| = |x-(2x-1)| = |-x+1|$. If $x > 1$ then $|-x+1| = x-1$. In conclusion, for all $x > 1$ (and *a fortiori* $x > 2$, we have $|x-|1-2x|| = x-1$.

1.5.3 If $x < -2$ then $1+x < -1$ and hence $|1+x| = -(1+x) = -1-x$. Thus $|1-|1+x|| = |1-(-1-x)| = |2+x|$. But since $x < -2$, $x+2 < 0$ and so $|2+x| = -2-x$. We conclude that $|1-|1+x|| = -2-x$.

1.5.7 Se tiene

$$|1-2x| < 3 \iff -3 < 1-2x < 3 \iff -4 < -2x < 2 \iff -1 < x < 2 \iff x \in]-1;2[,$$

en donde ha of recordarse que el dividir a una desigualdad por una cantidad negativa, se invierte el sentido of la desigualdad.

1.5.8 Four. Either $x^2-4x = -3$ or$x^2-4x = 3$. Thus $x \in \{1,3,2-\sqrt{7},2+\sqrt{7}\}$.

1.5.9 We know that

$$|x-3| = \pm(x-3) \qquad \text{and} \quad \text{that} \qquad |x+2| = \pm(x+2).$$

We puncture the real line at $x = 3$ and at $x = -2$, that is, where the absolute value terms change sign. We have

$x \in$	$]-\infty;-2[$	$]-2;3[$	$]3;+\infty[$
$\lvert x-3\rvert =$	$3-x$	$3-x$	$x-3$
$\lvert x+2\rvert =$	$-x-2$	$x+2$	$x+2$
$\lvert x-3\rvert+\lvert x+2\rvert =$	$-2x+1$	5	$2x-1$

Therefore ,

$$|x-3|+|x+2| = \begin{cases} -2x+1 & \text{if } x \leq -2, \\ 5 & \text{if } -2 \leq x \leq 3, \\ 2x-1 & \text{if } x \geq 3. \end{cases}$$

1. To solve $|x-3|+|x+2| = 3$, we need

   $$-2x+1 = 3 \quad \text{if } x \leq -2, \quad 5 = 3 \quad \text{if } -2 \leq x \leq 3, \quad 2x-1 = 3 \quad \text{if } x \geq 3.$$

   The first equation gives $x = -1$. As $-1 \notin]-\infty;-2]$, this is a spurious solution. The second equation is a contradiction. In the third equation, $2x-1 = 3 \implies x = 2$, which is also spurious since $2 \notin [3;+\infty[$. Therefore the equation $|x-3|+|x+2| = 3$ does not have real solutions.
2. To solve $|x-3|+|x+2| = 3$, we need

   $$-2x+1 = 5 \quad \text{if } x \leq -2, \quad 5 = 5 \quad \text{if } -2 \leq x \leq 3, \quad 2x-1 = 5 \quad \text{if } x \geq 3.$$

   The first equation gives $x = -2$. As $-2 \in]-\infty;-2]$, which is a legitimate solution. The second equation is a tautology, which means that all the elements in the interval $[-2;3]$ are solutions. In the third equation $2x-1 = 5 \implies x = 3$, which is also a legitimate solution, since $3 \in [3;+\infty[$. Therefore the equation $|x-3|+|x+2| = 5$ has an infinite number of real solutions, all in the interval $[-2;3]$.
3. To solve $|x-3|+|x+2| = 7$, we need

   $$-2x+1 = 7 \quad \text{if } x \leq -2, \quad 5 = 7 \quad \text{if } -2 \leq x \leq 3, \quad 2x-1 = 7 \quad \text{if } x \geq 3.$$

   The first equation gives $x = -3$. As $-3 \in]-\infty;-2]$, this is a legitimate solution. The second equation is a contradiction. In the third equation $2x-1 = 7 \implies x = 4$, which is also a legitimate solution since $4 \in [3;+\infty[$. Therefore the equation $|x-3|+|x+2| = 3$ has exactly two real solutions:

   $$\{x \in \mathbb{R} : |x-3|+|x+2| = 7\} = \{-3,4\}.$$

1.5.10 $x \in \{-2,1+\sqrt{5}\}$

1.5.11 We have

$$\begin{aligned} |5x-2| = |2x+1| \quad &\iff \quad (5x-2 = 2x+1) \text{ or } (5x-2 = -(2x+1)) \\ &\iff \quad (x = 1) \text{ or } (x = \frac{1}{7}) \\ &\iff \quad x \in \left\{\frac{1}{7},1\right\} \end{aligned}$$

1.5.12 The first term vanishes when $x = 2$ and the second term vanishes when $x = 3$. We decompose $\mathbb{R}$ into (overlapping) intervals with endpoints at the places where each of the expressions in absolute values vanish. Thus we have

$$\mathbb{R} =]-\infty;2] \cup [2;3] \cup [3;+\infty[.$$

We examine the sign diagram

$x \in$	$]-\infty;2]$	$[2;3]$	$[3;+\infty[$
$\lvert x-2\rvert =$	$-x+2$	$x-2$	$x-2$
$\lvert x-3\rvert =$	$-x+3$	$-x+3$	$x-3$
$\lvert x-2\rvert + \lvert x-3\rvert =$	$-2x+5$	1	$2x-5$

Thus on $]-\infty;2]$ we need $-2x+5 = 1$ from where $x = 2$. On $[2;3]$ we obtain the identity $1 = 1$. This means that all the numbers on this interval are solutions to this equation. On $[3;+\infty[$ we need $2x-5 = 1$ from where $x = 3$. Upon assembling all this, the solution set is $\{x : x \in [2;3]\}$.

1.5.13 $\{-\frac{1}{2}, \frac{3}{2}\}$

1.5.14 $\{x|x \in [0;1]\}$

1.5.15 $\{-1\}$

1.5.16 $[1;+\infty[$

1.5.17 $]-\infty;-2]$

1.5.19 $\{\frac{3}{2} + \frac{\sqrt{17}}{2}, \frac{3}{2} - \frac{\sqrt{17}}{2}, 1, 2\}$

1.5.20 $\{-1, 1\}$

1.5.21 $\{-3, -2, 2, 3\}$

1.5.22 $\{-6, 1, 2, 3\}$

1.5.23 We have

$$|x+3| = \begin{cases} -x-3 & \text{if } x+3 < 0, \\ x+3 & \text{if } x+3 \geq 0. \end{cases} \qquad |x-4| = \begin{cases} -x+4 & \text{if } x-4 < 0, \\ x-4 & \text{if } x-4 \geq 0. \end{cases}$$

This means that when $x < -3$

$$|x+3| - |x-4| = (-x-3) - (-x+4) = -7,$$

a constant. Since at $x = -3$ we also obtain -7, the result holds true for the larger interval $x \leq -3$.

1.5.24 There are four solutions: $\{-1-\sqrt{2}, -1+\sqrt{2}, 1-\sqrt{2}, -1+\sqrt{2}\}$.

1.5.25 $]-\infty;-1]$.

1.5.26 Clearly, $\max(x,y) + \min(x,y) = x+y$. Now, either $|x-y| = x-y$ and so $x \geq y$, which signifies that $\max(x,y) - \min(x,y) = x-y$, or $|x-y| = -(x-y) = y-x$, which means that $y \geq x$ and thus $\max(x,y) - \min(x,y) = y-x$. At any rate, $\max(x,y) - \min(x,y) = |x-y|$. Solving the system of equations

$$\begin{aligned} \max(x,y) + \min(x,y) &= x+y \\ \max(x,y) - \min(x,y) &= |x-y|, \end{aligned}$$

for $\max(x,y)$ and $\min(x,y)$, we obtain the result.

1.5.27 $\{x \in \mathbb{R} : |x-1||x+2| > 4\} =]-\infty;-3] \cup [2;+\infty[$

1.5.28 $]-4;-1[\cup]2;5[$

2.1.2 4.5 square units.

2.1.3 TFTFTTF

2.1.4 False. $(\mathbb{R}\setminus\{0\})^2$ consists of the plane minus the axes. $\mathbb{R}^2\setminus\{(0,0)\}$ consists of the plane minus the origin.

2.2.1 $2\sqrt{10}$

2.2.2 $\sqrt{2}|b-a|$

2.2.3 $\sqrt{(a^2-b)^2+(b^2-a)^2}$

2.2.4 We have,

$$\mathbf{d}\langle(a,b),\left(\frac{a+c}{2},\frac{b+d}{2}\right)\rangle=\sqrt{\left(a-\left(\frac{a+c}{2}\right)\right)^2+\left(b-\left(\frac{b+d}{2}\right)\right)^2}=\sqrt{\left(\frac{a-c}{2}\right)^2+\left(\frac{b-d}{2}\right)^2}.$$

Also,

$$\mathbf{d}\langle\left(\frac{a+c}{2},\frac{b+d}{2}\right),(c,d)\rangle=\sqrt{\left(\left(\frac{a+c}{2}\right)-c\right)^2+\left(\left(\frac{b+d}{2}\right)-d\right)^2}=\sqrt{\left(\frac{a-c}{2}\right)^2+\left(\frac{b-d}{2}\right)^2}.$$

2.2.5 $\sqrt{4x^2+(a+b)^2t^2}$

2.2.6 C is the point that divides AB in the ratio $3:2$. By Joachimstal's formula,

$$C=\left(\frac{2\cdot 1+3\cdot 4}{3+2},\frac{2\cdot 5+3\cdot 10}{3+2}\right)=\left(\frac{14}{5},8\right).$$

2.2.7 We have

$$\sqrt{(x-0)^2+(1-2)^2}=2\implies\sqrt{x^2+1}=2\implies x^2+1=4\implies x^2=3\implies x=\pm\sqrt{3}.$$

2.2.8 The bug should travel along two line segments: first from $(-1,1)$ the origin, and then from the origin to $(2,1)$ avoiding quadrant II altogether. For, if $a>0,b>0$ then the line segment joining $(-b,0)$ and $(a,0)$ lies in quadrant II, it is $\sqrt{a^2+b^2}$ long, and the bug spends an amount of time equal to $\dfrac{\sqrt{a^2+b^2}}{2}$ on this line. But a path on the axes from $(-b,0)$ to $(a,0)$ is $a+b$ units long and the bug spends an amount of time equal to $a+b$ there. Thus as long as

$$a+b\le\frac{a^2+b^2}{2}$$

the bug should avoid quadrant II completely. But by the Arithmetic-Mean-Geometric-Mean Inequality we have

$$2ab\le a^2+b^2\implies(a+b)^2=a^2+2ab+b^2\le 2a^2+2b^2\implies a+b\le\sqrt{2}\sqrt{a^2+b^2},$$

which means that as long as the speed of the bug in quadrant II is $<\dfrac{1}{\sqrt{2}}$ then the bug will better avoid quadrant II. Since $\dfrac{1}{2}<\dfrac{1}{\sqrt{2}}$, this follows in our case.

2.2.9 $(0,-3/4)$

2.2.10 $(2b-a,2a-b)$

2.2.11 Without loss of generality assume that the rectangle $ABCD$ has vertices at $A(0,0)$, $B(u,0)$, $C(u,v)$ and $D(0,v)$. Its diagonals are AC and BD, which results in

$$AC=\sqrt{(u-0)^2+(v-0)^2}=\sqrt{u^2+v^2},$$

and

$$BD=\sqrt{(u-0)^2+(0-v)^2}=\sqrt{u^2+v^2},$$

demonstrating their equality.

2.2.12 Without loss of generality, assume that the parallelogram $ABCD$ has vertices at $A(0,0)$, $B(u,0)$, $C(u+w,v)$ and $D(w,v)$. The coordinates of the midpoint of the segment AC are $\left(\dfrac{u+w}{2},\dfrac{v}{2}\right)$, which are the coordinates of the midpoint of BD, demonstrating the result.

2.2.13 Its x coordinate is

$$\frac{1}{2}-\frac{1}{8}+\frac{1}{32}-\cdots=\frac{\frac{1}{2}}{1-\frac{-1}{4}}=\frac{2}{5}.$$

Its y coordinate is

$$1-\frac{1}{4}+\frac{1}{16}-\cdots=\frac{1}{1-\frac{-1}{4}}=\frac{4}{5}.$$

Therefore, the fly ends up in

$$\left(\frac{2}{5},\frac{4}{5}\right).$$

Here we have used the fact the sum of an infinite geometric progression with common ratio r, with $|r|<1$ and first term a is

$$a+ar+ar^2+ar^3+\cdots=\frac{a}{1-r}.$$

2.2.14 $(\frac{3a+b}{4},\frac{3b+a}{4})$

2.2.15 (a,b) ; $(-a,-b)$; $(a,-b)$

2.2.16 It is enough to prove this in the case when a,b,c,d are all positive. To this end, put $O=(0,0)$, $L=(a,b)$ and $M=(a+c,b+d)$. By the triangle inequality $OM\leq OL+LM$, where equality occurs if and only if the points are collinear. But then

$$\sqrt{(a+c)^2+(b+d)^2}=OM\leq OL+LM=\sqrt{a^2+b^2}+\sqrt{c^2+d^2},$$

and equality occurs if and only if the points are collinear, that is $\dfrac{a}{b}=\dfrac{c}{d}$.

2.2.18 Use the above generalisation of Minkowski's Inequality and the fact that $17^2+144^2=145^2$. The desired value is S_{12}.

2.3.2 $(x-2)^2+(y-3)^2=2$

2.3.3 We must find the radius of this circle. Since the radius is the distance from the centre of the circle to any point on the circle, we see that the required radius is

$$\sqrt{(-1-1)^2+(1-2)^2}=\sqrt{5}\approx 2.236.$$

The equation sought is thus

$$(x+1)^2+(y-1)^2=5.$$

2.3.4 (1) $x^2+(y-1)^2=36, C=(0,1), R=6$. (2) $(x+2)^2+(y-1)^2=25, C=(-2,1), R=5$, (3) $(x+2)^2+(y-1)^2=10, C=(-2,1), R=\sqrt{10}$, (4) $(x-2)^2+y^2=12, C=(2,0), R=2\sqrt{3}$ (5) $(x+\frac{1}{2})^2+(y-\frac{3}{2})^2=\frac{5}{8}, C=(-\frac{1}{2},\frac{3}{2}), R=\sqrt{\frac{5}{8}}$, (6) $(x+\frac{1}{\sqrt{3}})^2+(y-\frac{\sqrt{3}}{3})^2=\frac{5}{3}, C=(-\frac{\sqrt{3}}{3},\sqrt{3}), R=\sqrt{\frac{5}{3}}$

2.3.6 $(x-1)^2+(y-3)^2=5$

2.3.7 $(x-\frac{9}{10})^2+(y-\frac{1}{10})^2=\frac{221}{50}$

2.3.9 This is asking to draw the circles $x^2+y^2=100$, $(x+4)^2+y^2=4$, $(x-4)^2+y^2=4$, $x^2+(y+4)^2=4$, all in the same set of axes.The picture appears in figure F.1.

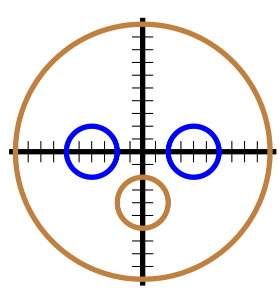


Figure F.1: Problem 2.3.9.

2.4.2 This is asking to draw the circle $x^2+y^2=100$, and the semicircles $y=\sqrt{4-(x+4)^2}$, $y=\sqrt{4-(x-4)^2}$, $y=-4-\sqrt{4-x^2}$, all in the same set of axes.The picture appears in figure F.1.

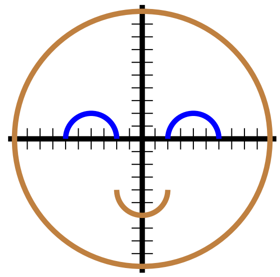


Figure F.2: Problem 2.4.2.

2.5.2 $\frac{7}{9}$

2.5.3 $-\frac{b}{a}$

2.5.4 -6

2.5.5 $y=\frac{x}{4}+\frac{1}{4}$

2.5.6 $y=-x+b+a$

2.5.7 $y=(a+b)x-ab$

2.5.8 $m=2$

2.5.10 Let $(x,0)$ be the coordinates of S. Since the slope of the line segment SM is 2, we have

$$\frac{2-0}{2-x}=2 \implies x=1,$$

whence S is the point $(1,0)$. Let $(a,0)$ be the coordinates of A. Since $SM=MA$, we have

$$\sqrt{(a-2)^2+(0-2)^2}=\sqrt{(1-2)^2+(0-2)^2} \implies (a-2)^2+4=5 \implies a\in\{1,3\}.$$

This means that A is the point $(3,0)$. Let B be point $(0,y)$. Since A,B,M are collinear, we may compute the slope in two different ways to obtain,

$$\frac{y-2}{0-2}=\frac{2-0}{2-3} \implies y-2=4 \implies y=6.$$

Thus B is the point $(0,6)$.

2.5.11 Let required point be (x,y). The distance of this point to its projection on the x-axis is $|y|$ and similarly, the distance of this point to its projection on the x-axis is $|x|$. We need

$$|y|=|x| \implies |6-2x|=|x| \implies 6-2x=x \qquad \text{or} \qquad 6-2x=-x.$$

The first case gives $x=2$ and the point is $(2,2)$, and the second case gives $x=6$ and the point is $(6,-6)$.

2.5.12 $x=3$

2.5.13 This is asking to graph the lines $x=-1$, $x=1$, $y=-1$, $y=1$, $y=-x$, and $y=x$, all on the same set of axes. The picture appears in figure F.3.

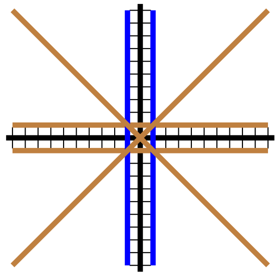


Figure F.3: Problem 2.5.13.

2.6.1 $y=4x-14$

2.6.2 $y=\dfrac{b}{a}x$

2.6.3 $y=-\dfrac{1}{4}x+\dfrac{29}{4}$

2.6.4 $y=-\dfrac{a}{b}x+b+\dfrac{a^2}{b}$

2.6.5 $y=\frac{3}{4}x-9$

2.6.6 $y=-\frac{4}{3}x+16$

2.6.7 Notice that there is a radius of the circle connecting $(0,0)$ and $(\frac{1}{2},\frac{\sqrt{3}}{2})$. The line passing through these two points is $y=\sqrt{3}x$. Hence, since the tangent line is perpendicular to the radius at the point of tangency, the line sought is of the form $y=-\dfrac{\sqrt{3}}{3}x+k$. To find k observe that $\dfrac{\sqrt{3}}{2}=-\dfrac{3}{4}+k \implies k=\dfrac{3\sqrt{3}}{4}$. Finally, the desired line is $y=-\dfrac{\sqrt{3}}{3}x+\dfrac{3\sqrt{3}}{4}$.

2.6.8 $L_1: y=(a+b)x+1-a-b$, $L_2: y=-\dfrac{x}{a+b}+\dfrac{a+b+1}{a+b}$

2.6.9 (1) $t=4/3$, (2) $t=6/7$, (3) $t=1/2$, (4) $t=3$, (5) $t=-7$, (6) $t=7/9$, (7) $t=7/4$, (8) $(3,-1)$

2.6.10 We have

1. If L_t passes through $(-2,3)$ then
$$(t-2)(-2)+(t+3)(3)+10t-5=0,$$
from where $t=-\frac{8}{11}$. In this case the line is
$$-\frac{30}{11}x+\frac{25}{11}y-\frac{135}{11}=0.$$
2. L_t will be parallel to the x-axis if the x-term disappears, which necessitates $t-2=0$ or $t=2$. In this case the line is
$$y=-3.$$
3. L_t will be parallel to the y-axis if the y-term disappears, which necessitates $t+3=0$ or $t=-3$. In this case the line is
$$x=-7.$$
4. The line $x-2y-6=0$ has gradient $\frac{1}{2}$ and L_t has gradient $\frac{2-t}{t+3}$. The lines will be parallel when $\frac{2-t}{t+3}=\frac{1}{2}$ or $t=1/3$. In this case the line is
$$-\frac{5}{3}x+\frac{10}{3}y-\frac{5}{3}=0.$$
5. The line $y=-\frac{1}{4}x-5$ has gradient $-\frac{1}{4}$ and L_t has gradient $\frac{2-t}{t+3}$. The lines will be perpendicular when $\frac{2-t}{t+3}=4$ or $t=-2$. In this case the line is
$$-4x+y-25=0.$$

6. If such a point existed, it would pass through the horizontal and vertical lines found above, thus $(x_0,y_0)=(-7,-3)$ is a candidate for the point sought. That $(-7,-3)$ passes through every line L_t, no matter the choice of t is seen from

$$(t-2)(-7)+(t+3)(-3)+10t-5=-7t+14-3t-9+10t-5=0.$$

2.6.12 $\sqrt{2}$

2.6.13 $\sqrt{1+a^2}$

2.6.14 The radius of the circle is the distance from the centre to the tangent line. This radius is then

$$r=\left|\frac{3-2\cdot 4+3}{\sqrt{1^2+(-2)^2}}\right|=\frac{2}{\sqrt{5}}.$$

The desired equation is

$$(x-3)^2+(y-4)^2=\frac{4}{5}.$$

2.6.15 Let $M_A\left(\frac{b}{2},\frac{c}{2}\right)$, $M_B\left(\frac{a}{2},\frac{c}{2}\right)$, $M_C\left(\frac{a+b}{2},0\right)$ be the respective midpoints of the sides BC, CA and AB. The equations of the straight line containing the medians are then

$$\overleftrightarrow{AM_A}:\quad y=\frac{\frac{c}{2}}{\frac{b}{2}-a}x+\frac{ca}{2a-b}=\frac{c}{b-2a}x+\frac{ca}{2a-b},$$

$$\overleftrightarrow{BM_B}:\quad y=\frac{\frac{c}{2}}{\frac{a}{2}-b}x+\frac{cb}{2b-a}=\frac{c}{a-2b}x+\frac{cb}{2b-a},$$

and

$$\overleftrightarrow{CM_C}:\quad y=\frac{c}{-\frac{a+b}{2}}x+c=-\frac{2c}{a+b}x+c.$$

Since we are supposing the triangle to be non-degenerate (that is, it isn't "flat"), $\overleftrightarrow{AM_A}$ and $\overleftrightarrow{BM_B}$ must intersect. Then

$$\frac{c}{b-2a}x+\frac{ca}{2a-b}=\frac{c}{a-2b}x+\frac{cb}{2b-a}\implies x=\frac{a+b}{3}.$$

To find the value of the coordinate y, we substitute $x=\dfrac{a+b}{3}$ in any of these three lines, say the first:

$$y=\frac{c}{b-2a}x+\frac{ca}{2a-b}=\frac{c}{b-2a}\cdot\frac{a+b}{3}+\frac{ca}{2a-b}=\frac{c}{3}.$$

To conclude, we must shew that the point $\left(\dfrac{a+b}{3},\dfrac{c}{3}\right)$ lies on the line $\overleftrightarrow{CM_C}$, that is, we must verify that

$$\frac{c}{3}\stackrel{?}{=}-\frac{2c}{a+b}\cdot\frac{a+b}{3}+c,$$

which we leave to the reader.

2.6.16 Let H_A, H_B and H_C be the feet of the altitudes from A to BC, from B to CA and from C to AB, respectively. As the altitudes are perpendicular to the sides, the respective slopes of $\overleftrightarrow{AH_A}$, $\overleftrightarrow{BH_B}$, $\overleftrightarrow{CH_C}$, will be the opposite of the reciprocals of the slopes of the straight lines BC, CA, AB. We then find the equations

$$\overleftrightarrow{AM_A}:\quad y=\frac{b}{c}x-\frac{ab}{c},$$

$$\overleftrightarrow{BM_B}:\quad y=\frac{a}{c}x-\frac{ab}{c},$$

and

$$\overleftrightarrow{CM_C}:\quad x=0.$$

Since we are supposing the triangle to be non-degenerate (that is, it isn't "flat"), $\overleftrightarrow{AM_A}$, $\overleftrightarrow{BM_B}$ must intersect. Hence $x=0$ and

$$\frac{b}{c}x-\frac{ab}{c}=\frac{a}{c}x-\frac{ab}{c}\implies x=0,$$

and since the triangle is non-degenerate, $a\neq b$. Hence, $y=-\dfrac{ab}{c}$. Obviously $\left(0,-\dfrac{ab}{c}\right)$ is also on L_{CM_C}, demonstrating the result.

2.6.17 Let $\overleftrightarrow{A'B'}$, $\overleftrightarrow{B'C'}$, $\overleftrightarrow{C'A'}$ be the perpendicular bisectors to AB, BC and CA. Then

$$\overleftrightarrow{A'B'}: \quad x = \frac{a+b}{2},$$

$$\overleftrightarrow{B'C'}: \quad y = \frac{b}{c}x,$$

$$\overleftrightarrow{C'A'}: \quad y = \frac{a}{c}x.$$

Since $\overleftrightarrow{A'B'}$, $\overleftrightarrow{B'C'}$ must intersect, $x = \dfrac{a+b}{2}$ and

$$y = \frac{b(a+b)}{2c}.$$

Since $\left(\dfrac{a+b}{2}, \dfrac{ab+c^2}{2c}\right)$ also lies on $\overleftrightarrow{C'A'}$, the result is obtained.

2.6.18 Suppose, without loss of generality, that the square $ABCD$ has vertices at $A(0,0)$, $B(a,0)$, $C(a,a)$ and $D(0,a)$. The slopes of the straight lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are 1 and -1, from where the result is achieved.

2.7.1

1. This is $y = (-x+1)-(-x)+(-x-1) = -x$.
2. This is $y = (-x+1)-(-x)+(x+1) = x+2$.
3. This is $y = (-x+1)-(+x)+(x+1) = -x+2$.
4. This is $y = (x-1)-(+x)+(x+1) = x$.
5. The graph of $\mathscr{C}$ appears in figure F.4.

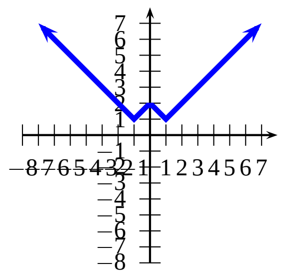


Figure F.4: Problem 2.7.1.

2.7.2 The graph appears in figure F.5.

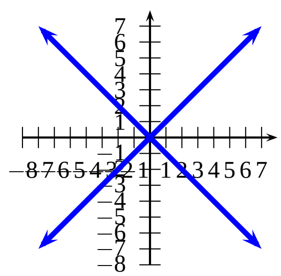


Figure F.5: Problem 2.7.2.

2.7.3 This is the curve

$$y = \frac{|x| + x}{2} = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The graph appears in figure F.6.

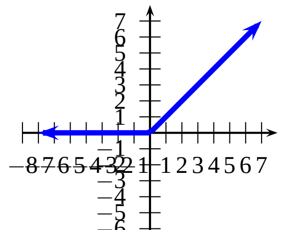


Figure F.6: Problem 2.7.3.

2.7.4 The set is composed of four segments of a circle inside the circle of equation $x^2 + y^2 = 16$ and bounded by the lines of equation $y = x + 4$, $y = x - 4$, $y = -x + 4$ and $y = -x - 4$. The graph appears in figure F.7.

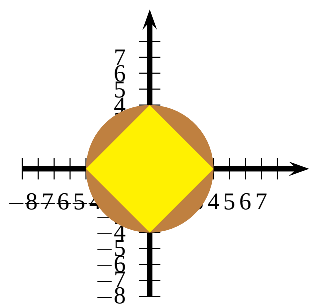


Figure F.7: Problem 2.7.4.

2.8.2 By the preceding exercise the focus is $(\frac{1}{4}, 0)$ and the directrix is $x = -\frac{1}{4}$.

2.8.3 If (x, y) is an arbitrary point on this parabola we must have

$$\frac{|-x-y|}{\sqrt{1+(-1)^2}} = \sqrt{(x-1)^2 + (y-1)^2}$$

Squaring and rearranging, the desired equation is

$$x^2 + y^2 - 2xy - 4x - 4y + 4 = 0.$$

2.8.5 The distance of (x, y) to $(2, 3)$ is $\sqrt{(x-2)^2 + (y-3)^2}$. The distance of (x, y) to the line $x = -4$ is $|x - (-4)| = |x + 4|$. We need

$$\sqrt{(x-2)^2 + (y-3)^2} = |x+4| \iff (x-2)^2 + (y-3)^2 = (x+4)^2 \iff x = \frac{y^2}{12} - \frac{y}{2} - \frac{1}{2},$$

from where the desired curve is a parabola.

2.8.6 Put $x > 0$ and $A = (0, 0)$, $B = \left(x, \frac{x^2}{2}\right)$ y $C = \left(-x, \frac{x^2}{2}\right)$. Then

$$AB = BC \implies \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 0\right)^2} = \sqrt{(x-(-x))^2 + \left(\frac{x^2}{2} - \frac{x^2}{2}\right)^2} \implies x^2\left(1 + \frac{x^2}{4}\right) = 4x^2 \implies x = 2\sqrt{3}.$$

The points are $A(0, 0)$, $B(2\sqrt{3}, 6)$ and $C(-2\sqrt{3}, 6)$.

3.1.1 We have

$$f(0)+f(1)+f(2)=\frac{0-1}{0^2+1}+\frac{1-1}{1^2+1}+\frac{2-1}{2^2+1}=-1+0+\frac{1}{5}=-\frac{4}{5}.$$

Also,

$$f(0+1+2)=f(3)=\frac{3-1}{3^2+1}=\frac{1}{5}.$$

Clearly then $f(0)+f(1)+f(2)\neq f(0+1+2)$.

Now,

$$f(x)=\frac{1}{x}\implies x^2-x=x^2+1\implies x=-1.$$

Also,

$$f(x)=x\implies x-1=x^3+x\implies x^3=-1\implies x=-1.$$

3.1.2 There are $2^3=8$ such functions:

1. f_1 given by $f_1(0)=f_1(1)=f_1(2)=-1$
2. f_2 given by $f_2(0)=1, f_2(1)=f_2(2)=-1$
3. f_3 given by $f_3(0)=f_3(1)=-1, f_3(2)=1$
4. f_4 given by $f_4(0)=-1, f_4(1)=1, f_4(2)=-1$
5. f_5 given by $f_5(0)=f_5(1)=f_5(2)=1$
6. f_6 given by $f_6(0)=-1, f_6(1)=f_6(2)=1$
7. f_7 given by $f_7(0)=f_7(1)=1, f_7(2)=-1$
8. f_8 given by $f_8(0)=1, f_8(1)=-1, f_8(2)=1$

3.1.3 There are $3^2=9$ such functions:

1. f_1 given by $f_1(-1)=f_1(1)=0$
2. f_2 given by $f_2(-1)=f_2(1)=1$
3. f_3 given by $f_3(-1)=f_3(1)=2$
4. f_4 given by $f_4(-1)=0, f_4(1)=1$
5. f_5 given by $f_5(-1)=0, f_5(1)=2$
6. f_6 given by $f_6(-1)=1, f_6(1)=2$
7. f_7 given by $f_7(-1)=1, f_7(1)=0$
8. f_8 given by $f_8(-1)=2, f_8(1)=0$
9. f_9 given by $f_9(-1)=2, f_9(1)=1$

3.1.4 $4x-2$

3.1.5 $6x^2+2h^2-6$

3.1.6

1. True. $f\left(\frac{a}{b}\right)=\frac{1}{\frac{a}{b}}=\frac{b}{a}=\frac{1}{a}\cdot\frac{1}{\frac{1}{b}}=\frac{f(a)}{f(b)}$.
2. False. For example, $f(1+1)=f(2)=\frac{1}{2}$, but $f(1)+f(1)=\frac{1}{1}+\frac{1}{1}=2$.
3. True. $f(a^2)=\frac{1}{a^2}=\left(\frac{1}{a}\right)^2=(f(a))^2$.

3.1.7 $a(3)=6$; x^2+x-6; $24-11x-10x^2+2x^3+x^4$

3.1.8 7, x^2-2x-1, x^4-4x^3+8x+2

3.1.9 We must look for all $x \in \mathbf{Dom}(f)$ such that $s(x) = x$. Thus

$$\begin{aligned} s(x) = x \quad & \implies \quad x^5 - 2x^3 + 2x = x \\ & \implies \quad x^5 - 2x^3 + x = 0 \\ & \implies \quad x(x^4 - 2x^2 + 1) = 0 \\ & \implies \quad x(x^2 - 1)^2 = 0 \\ & \implies \quad x(x+1)^2(x-1)^2 = 0. \end{aligned}$$

The solutions to this last equation are $\{-1, 0, 1\}$. Since $-1 \notin \mathbf{Dom}(s)$, the only fixed points of s are $x = 0$ and $x = 1$.

3.1.10 $h(x-1) = -11 + 7x - x^2$; $h(x) = -5 + 5x - x^2$; $h(x+1) = -1 + 3x - x^2$

3.1.11 $f(x) = x^2 - 2x + 1$; $f(x+2) = x^2 + 2x + 1$; $f(x-2) = x^2 - 6x + 9$

3.1.12 Rename the independent variable, say $h(1-s) = 2s$. Now, if $1 - s = 3x$ then $s = 1 - 3x$. Hence

$$h(3x) = h(1-s) = 2s = 2(1-3x) = 2 - 6x.$$

3.1.13 Consider the function $p : \mathbb{R} \to \mathbb{R}$, with

$$p(x) = (1 - x^2 + x^4)^{2003} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{8012} x^{8012}.$$

Then

1. $a_0 = p(0) = (1 - 0^2 + 0^4)^{2003} = 1.$
2. $a_0 + a_1 + a_2 + \cdots + a_{8012} = p(1) = (1 - 1^2 + 1^4)^{2003} = 1.$
3. $$\begin{aligned} a_0 - a_1 + a_2 - a_3 + \cdots - a_{8011} + a_{8012} &= p(-1) \\ &= (1 - (-1)^2 + (-1)^4)^{2003} \\ &= 1. \end{aligned}$$
4. The required sum is $\dfrac{p(1) + p(-1)}{2} = 1.$
5. The required sum is $\dfrac{p(1) - p(-1)}{2} = 0.$

3.1.14 We have

$$\begin{aligned} \left[f\left(\tfrac{27 + y^3}{y^3} \right) \right]^{\sqrt[3]{\frac{27}{y}}} &= \left[f\left(\left(\tfrac{3}{y}\right)^3 + 1 \right) \right]^{3\sqrt{\frac{3}{y}}} \\ &= \left(\left[f\left(\left(\tfrac{3}{y}\right)^3 + 1 \right) \right]^{\sqrt{\frac{3}{y}}} \right)^3 \\ &= 5^3 \\ &= 125. \end{aligned}$$

3.2.1

1. $d(-2) \in [2;3]$.
2. $d(-3)$ is undefined.
3. $d(0)$ is undefined.
4. $d(100) = 4$.

3.2.2 The graph appears in figure F.8.

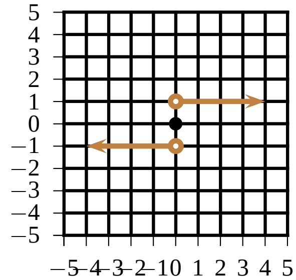


Figure F.8: Problem 3.2.2: d.

3.2.3 $\mathbf{Dom}(\mathbf{Id}) = \mathbb{R}$ and $\mathbf{Im}(\mathbf{Id}) = \mathbb{R}$.

3.2.4 $\mathbf{Dom}(\mathbf{AbsVal}) = \mathbb{R}$ and $\mathbf{Im}(\mathbf{AbsVal}) = [0;+\infty[$.

3.2.5 $\mathbf{Dom}(\mathbf{Sq}) = \mathbb{R}$ and $\mathbf{Im}(\mathbf{Sq}) = [0;+\infty[$.

3.2.6 $\mathbf{Dom}(\mathbf{Rt}) = [0;+\infty[$ and $\mathbf{Im}(\mathbf{Rt}) = [0;+\infty[$.

3.2.7 $\mathbf{Dom}(\mathbf{Sc}) = [-1;1]$ and $\mathbf{Im}(\mathbf{Sc}) = [0;1]$.

3.2.8 $\mathbf{Dom}(\mathbf{Rec}) = \mathbb{R} \setminus \{0\}$ and $\mathbf{Im}(\mathbf{Rec}) = \mathbb{R} \setminus \{0\}$.

3.2.9 The graph appears in figure F.9.

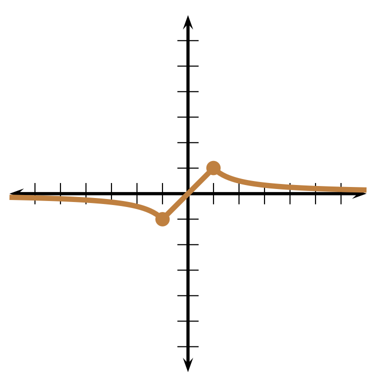


Figure F.9: Problem 3.2.9.

3.2.10 The first line segment $\mathscr{L}_1$ has slope

$$\text{slope } \mathscr{L}_1 = \frac{1-(-3)}{-1-(-4)} = \frac{4}{3},$$

and so the equation of the line containing this line segment is of the form $y = \frac{4}{3}x + k_1$. Since $(-1,1)$ is on the line, $1 = -\frac{4}{3} + k_1 \implies k_1 = \frac{7}{3}$, so this line segment is contained in the line $y = \frac{4}{3}x + \frac{7}{3}$. The second line segment $\mathscr{L}_2$ has slope

$$\text{slope } \mathscr{L}_2 = \frac{1-1}{2-(-1)} = 0,$$

and so this line segment is contained in the line $y = 1$. Finally, the third line segment $\mathscr{L}_3$ has slope

$$\text{slope } \mathscr{L}_3 = \frac{-5-1}{4-2} = -3,$$

and so this line segment is part of the line of the form $y = -3x + k_2$. Since $(1,2)$ is on the line, we have $2 = -3 + k_2 \implies k_2 = 5$, and so the line segment is contained on the line $y = -3x + 5$. Upon assembling all this we see that the piecewise function required is

$$f(x) = \begin{cases} \frac{4}{3}x + \frac{7}{3} & \text{if } x \in [-4;-1] \\ 1 & \text{if } x \in [-1;2] \\ -3x+5 & \text{if } x \in [2;4] \end{cases}$$

3.3.2
1. $\mathbb{R}$
2. $[-5;5]$
3. $\mathbb{R}$
4. $\mathbb{R}$
5. $]-\infty; 1-\sqrt{3}[\cup]1+\sqrt{3};+\infty[$
6. $\mathbb{R}$
7. $]-\infty;-1[\cup]-1;0]$
8. $]-1;1[$
9. $\{0\}$

3.3.4 $[-2\sqrt{3};0]\cup[2\sqrt{3};+\infty[$.

3.3.5 $x \in]-\infty;1-\sqrt{3}[\cup]1+\sqrt{3};+\infty[$.

3.3.6 They are

1. $\{-1\}$
2. $\varnothing$
3. $[0;2[\cup]3;+\infty[$
4. $]3;+\infty[$
5. $]-\infty;-3[\cup]-2;0[\cup]0;2[\cup]3;+\infty[$
6. $]-3;-2[\cup]0;2[\cup]3;+\infty[$
7. $\mathbb{R}\setminus\{-3,-2,2,3\}$

3.4.1
1. $[-4;2]$
2. $[-4;2]$
3. $]-4;2]$
4. $[-4;-2[\cup]-2;2[$
5. -2
6. 0
7. 0
8. undefined
9. 5
10. $\frac{1}{5}$

3.4.2 1)$\{-4,-2,0,2,4\}$ 2) $\{0,1,4\}$ 3) $\{0,1\}$. 4) $\{0,2\}$.

3.4.3 (1) 13, (2) 5981, (3) 10, (4) 1995

3.4.4 Observe that $a+b=f(1)=8$. We have $f(50)=50a+b$, $g(50)=50b+a$ and

$$f(g(50))=f(50b+a)=50ab+a^2+b, \qquad g(f(50))=g(50a+b)=50ab+b^2+a,$$

whence

$$28=f(g(50))-g(f(50))=a^2-b^2-(a-b)=(a-b)(a+b-1)=7(a-b) \implies a-b=4.$$

Therefore,

$$ab=\frac{(a+b)^2-(a-b)^2}{4}=\frac{64-16}{4}=12.$$

3.4.6 1) $[0;+\infty]$ 2) $[0;2]$ 3) $\{0\}$. 4) $[2;6]$. 5) $\sqrt{\sqrt{4-x^2}-2}$. 6) $\sqrt{6-x}$.

3.4.7 1) $[0;\sqrt{2}]$ 2) $]-\infty;0]$ 3) $[-2;0]$. 4) $\{-\sqrt{2},\sqrt{2}\}$. 5) $\sqrt{2+x}$. 6) $-\sqrt{-\sqrt{2-x^2}}$.

3.4.9 $\dfrac{\sqrt{2}}{2}$

3.4.10 $\dfrac{8}{4+x}$

3.4.11 $x=1/3$.

3.4.12 $(f\circ f)(x)=4x^2-4x^3+x^4$.

3.4.13 $c=-3$

3.4.14 If $y=0$ then $f(x+g(0))=2x+5$. Hence

$$f(x)=f(x-g(0)+g(0))2(x-g(0)+5)=2x-2g(0)+5.$$

We deduce that $f(0)=-2g(0)+5$ and hence,

$$-2g(0)+5=f(-g(y)+g(y))=2(-g(y))+y+5 \implies g(y)=g(0)+\frac{y}{2}.$$

This gives

$$g(x+f(y))=g(0)+\frac{x+f(y)}{2}=g(0)+\frac{x+2y-2g(0)+5}{2}=\frac{x+2y+5}{2}.$$

3.5.1 We have $f^{[2]}(x)=f(x+1)=(x+1)+1=x+2$, $f^{[3]}(x)=f(x+2)=(x+2)+1=x+3$ and so, recursively, $f^{[n]}(x)=x+n$.

3.5.2 We have $f^{[2]}(x)=f(2x)=2^2x$, $f^{[3]}(x)=f(2^2x)=2^3x$ and so, recursively, $f^{[n]}(x)=2^nx$.

3.5.3 Let $x=1$. Then $f(y)=yf(1)$. Since $f(1)$ is a constant, we may let $k=f(1)$. So all the functions satisfying the above equation satisfy $f(y)=ky$.

3.5.4 From $f(x)+2f(\frac{1}{x})=x$ we obtain $f\left(\frac{1}{x}\right)=\frac{x}{2}-\frac{1}{2}f(x)$. Also, substituting $1/x$ for x on the original equation we get

$$f(1/x)+2f(x)=1/x.$$

Hence

$$f(x)=\frac{1}{2x}-\frac{1}{2}f(1/x)=\frac{1}{2x}-\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}f(x)\right),$$

which yields $f(x)=\dfrac{2}{3x}-\dfrac{x}{3}$.

3.5.5 We have

$$(f(x))^2\cdot f\left(\frac{1-x}{1+x}\right)=64x,$$

whence

$$(f(x))^4\cdot\left(f\left(\frac{1-x}{1+x}\right)\right)^2=64^2x^2 \qquad (I)$$

Substitute x by $\frac{1-x}{1+x}$. Then

$$f\left(\frac{1-x}{1+x}\right)^2f(x)=64\left(\frac{1-x}{1+x}\right). \qquad (II)$$

Divide (I) by (II),

$$f(x)^3=64x^2\left(\frac{1+x}{1-x}\right),$$

from where the result follows.

3.5.8 We have

$$\begin{array}{lllll}
f(2) & = & (-1)^2 1 - 2f(1) & = & 1 - 2f(1) \\
f(3) & = & (-1)^3 2 - 2f(2) & = & -2 - 2f(2) \\
f(4) & = & (-1)^4 3 - 2f(3) & = & 3 - 2f(3) \\
f(5) & = & (-1)^5 4 - 2f(4) & = & -4 - 2f(4) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
f(999) & = & (-1)^{999} 998 - 2f(998) & = & -998 - 2f(998) \\
f(1000) & = & (-1)^{1000} 999 - 2f(999) & = & 999 - 2f(999) \\
f(1001) & = & (-1)^{1001} 1000 - 2f(1000) & = & -1000 - 2f(1000)
\end{array}$$

Adding columnwise,

$$f(2)+f(3)+\cdots+f(1001) = 1-2+3-\cdots+999-1000-2(f(1)+f(2)+\cdot+f(1000)).$$

This gives

$$2f(1)+3(f(2)+f(3)+\cdots+f(1000))+f(1001) = -500.$$

Since $f(1) = f(1001)$ we have $2f(1)+f(1001) = 3f(1)$. Therefore

$$f(1)+f(2)+\cdots+f(1000) = -\frac{500}{3}.$$

3.5.9 1

3.5.10 Set $a = b = 0$. Then $(f(0))^2 = f(0)f(0) = f(0+0) = f(0)$. This gives $f(0)^2 = f(0)$. Since $f(0) > 0$ we can divide both sides of this equality to get $f(0) = 1$.

Further, set $b = -a$. Then $f(a)f(-a) = f(a-a) = f(0) = 1$. Since $f(a) \neq 0$, may divide by $f(a)$ to obtain $f(-a) = \dfrac{1}{f(a)}$.
Finally taking $a = b$ we obtain $(f(a))^2 = f(a)f(a) = f(a+a) = f(2a)$. Hence $f(2a) = (f(a))^2$

3.6.1 Assume $g(s_1) = g(s_2)$. Then

$$\begin{array}{rrcl}
g(s_1) = g(s_2) & \implies 2s_1 + 1 & = & 2s_2 + 1 \\
& \implies 2s_1 & = & 2s_2 \\
& \implies s_1 & = & s_2
\end{array}$$

We have shewn that $g(s_1) = g(s_2) \implies s_1 = s_2$, and the function is thus injective.

To prove that g is surjective, we must prove that $(\forall\, b \in \mathbb{R})\ (\exists a)$ such that $g(a) = b$. We choose a so that $a = \dfrac{b-1}{2}$. Then

$$g(a) = g\left(\frac{b-1}{2}\right) = 2\left(\frac{b-1}{2}\right) + 1 = b - 1 + 1 = b.$$

Our choice of a works and hence the function is surjective.

3.6.4 We must shew that there is a solution x for the equation $f(x) = b, b \in \mathbb{R} \setminus \{2\}$. Now

$$f(x) = b \implies \frac{2x}{x+1} = b \implies x = \frac{b}{2-b}.$$

Thus as long as $b \neq 2$ there is $x \in \mathbb{R}$ with $f(x) = b$. Since there is no x such that $g(x) = 2$ and $2 \in$ **Target** (g), g is not surjective.

3.6.5

1. neither, $f(-1) = f(1)$ so not injective. There is no a with $f(a) = -1$, so not surjective.
2. surjective, $f(1) = f(-1)$ so not injective.

3. surjective, not injective.
4. injective, as proved in text, there is no a with $f(a) = -1$, so not surjective.
5. neither, $|1| = |-1|$ so not injective, there is no a with $|a| = -1$, so not surjective.
6. injective, non-surjective since, say, there is no a with $-|a| = 1$.
7. surjective, non-injective since, say, $|-1| = |1|$ but $-1 \neq 1$.
8. bijective.

3.7.1 Since $c(c^{-1}(x)) = x$, we have $\dfrac{c^{-1}(x)}{c^{-1}(x)+2} = x$. Solving for $c^{-1}(x)$ we obtain $c^{-1}(x) = \dfrac{2x}{1-x} = -2 + \dfrac{2}{1-x}$. The inverse of c is therefore

$$c^{-1}: \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R}\setminus\{-2\} \\ x & \mapsto & -2+\dfrac{2}{1-x} \end{array}.$$

3.7.2 $f^{-1}: \mathbb{R} \to reals$, $f^{-1}(x) = \sqrt[3]{\dfrac{x-}{2}}$

3.7.3 $f: \mathbb{R}\setminus\{1\} \to \mathbb{R}\setminus\{1\}$, $f^{-1}(x) = \dfrac{x^3+2}{x^3-1}$.

3.7.4 $(f \circ g)^{-1}(1) = (g^{-1} \circ f^{-1})(1) = g^{-1}(f^{-1}(1)) = g^{-1}(3) = 2$.

3.7.5 Since $x^2 - 4x + 5 = (x-2)^2 + 1$, consider $I_1 =]-\infty; 2]$ and $I_2 = [2; +\infty[$.

3.7.6

1. The first piece of f is a line segment with endpoints at $(-5,5), (0,-1)$, and whose slope is $-\dfrac{6}{5}$. Thus the equation of f os $f(x) = -\dfrac{6}{5}x - 1$. Putting $y = -\dfrac{6}{5}x - 1$ and solving for x we obtain $x = -\dfrac{5}{6}y - \dfrac{5}{6}$. We deduce that $f^{-1}(x) = -\dfrac{5}{6}x - \dfrac{5}{6}$. For $x \in [-5;0]$, $-1 \leq f(x) \leq 5$, and hence the formula for f^{-1} is only valid for $-1 \leq x \leq 5$.
2. The second piece is a line segment with endpoints at $(0,-1), (5,-3)$, which has slope $-\dfrac{2}{5}$. The equation of f is $f(x) = -\dfrac{2}{5}x - 1$. Putting $y = -\dfrac{2}{5}x - 1$ and solving for x, we obtain $x = -\dfrac{5}{2}y - \dfrac{5}{2}$. We deduce that $f^{-1}(x) = -\dfrac{5}{2}x - \dfrac{5}{2}$. For $x \in [0;5]$, $-3 \leq f(x) \leq -1$, and so this formula for f^{-1} is only valid for $-3 \leq x \leq -1$.
3. The graph of f^{-1} appears in figure F.10.

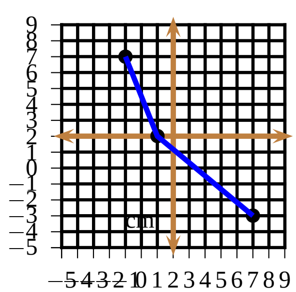


Figure F.10: Problem 3.7.6.

3.7.7 We have

1. The expression under the cubic root must not be 0. Hence $x^5 \neq 1$ and the natural domain is $\mathbb{R}\setminus\{1\}$.

2. Put
$$y = \frac{1}{\sqrt[3]{x^5-1}}.$$
Now exchange x and y and solve for y:
$$x = \frac{1}{\sqrt[3]{y^5-1}} \implies x^3(y^5-1) = 1 \implies y = \sqrt[5]{\frac{x^3+1}{x^3}}.$$
Hence
$$f^{-1}(x) = \sqrt[5]{\frac{x^3+1}{x^3}}.$$
3. As x varies in $\mathbb{R}\setminus\{1\}$, the expression $\dfrac{1}{\sqrt[3]{x^5-1}}$ assumes all positive and negative values, but it is never 0. Thus $\mathbf{Im}(f) = \mathbb{R}\setminus\{0\}$. The expression for $f^{-1}(x)$ is undefined when $x = 0$. Hence the natural domain of f^{-1} is $\mathbb{R}\setminus\{0\}$.
4. The function
$$f: \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R}\setminus\{0\} \\ x & \mapsto & \dfrac{1}{\sqrt[3]{x^5-1}} \end{array}$$
is a bijection with inverse
$$f^{-1}: \begin{array}{ccc} \mathbb{R}\setminus\{0\} & \to & \mathbb{R}\setminus\{1\} \\ x & \mapsto & \sqrt[5]{\dfrac{x^3+1}{x^3}} \end{array}.$$

3.7.8 Since $x \geq 0$, $f(x) = x^2 - \dfrac{1}{4}$ has inverse $f^{-1}(x) = \sqrt{x + \dfrac{1}{4}}$. The graphs of f and f^{-1} meet on the line $y = x$. Hence we are looking for a positive solution to
$$x^2 - \frac{1}{4} = x \implies x = \frac{1+\sqrt{2}}{2}.$$

3.7.9 1) Yes, f is a bijection. $f^{-1}(f(h(4))) = h(4) = 1$, 2) No

3.7.10 3

3.7.11 $t^{-1}: \begin{array}{ccc} [0;+\infty[& \to &]-\infty;1] \\ x & \mapsto & 1-x^2 \end{array}$

3.7.12 Either $a = 1, b = 0$ or $a = -1$ and b arbitrary.

3.7.18 We have $f^{-1}: \mathbb{R} \to \mathbb{R}$, with
$$f^{-1}(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$
The graph of f^{-1} appears in figure F.11.

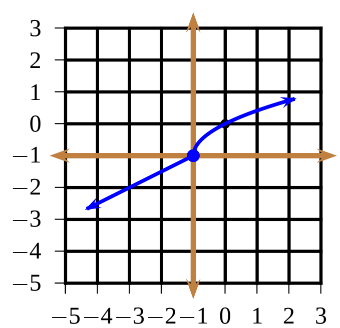


Figure F.11: Problem 3.7.18.

3.7.20 The inverse of

$$f: \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}$$

is

$$f^{-1}: \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & \sqrt{x} \end{array}.$$

In diagram F.12, each rectangle V_k has its lower left corner at $(0, \dfrac{k}{10})$, base $\sqrt{\frac{k}{10}}$ and height $\frac{1}{10}$. Each rectangle H_k has lower left corner at $(\dfrac{k}{10}, 0)$, base $\frac{1}{10}$ and height $(\frac{k}{10})^2$. The collective area of these rectangles is

$$\frac{1}{10}\left(\left(\frac{1}{10}\right)^2+\sqrt{\frac{1}{10}}+\left(\frac{2}{10}\right)^2+\sqrt{\frac{2}{10}}+\left(\frac{3}{10}\right)^2+\sqrt{\frac{3}{10}}+\cdots+\left(\frac{9}{10}\right)^2+\sqrt{\frac{9}{10}}\right)$$

Since these grey rectangles do not intersect with the green squares on the corners, their collective area is less than the area of the unit square minus these smaller squares: $1-\dfrac{1}{100}-\dfrac{4}{100}=\dfrac{95}{100}$. We thus conclude that

$$\frac{1}{10}\left(\left(\frac{1}{10}\right)^2+\sqrt{\frac{1}{10}}+\left(\frac{2}{10}\right)^2+\sqrt{\frac{2}{10}}+\left(\frac{3}{10}\right)^2+\sqrt{\frac{3}{10}}+\cdots+\left(\frac{9}{10}\right)^2+\sqrt{\frac{9}{10}}\right)<\frac{95}{100}.$$

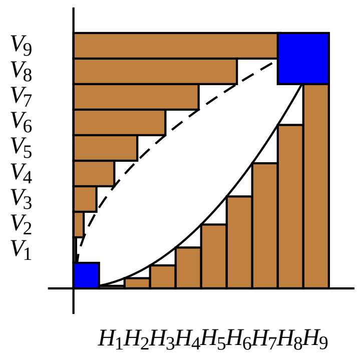


Figure F.12: Problem 3.7.20.

4.1.2 $y = f(x-2)-1 = (x-2)^2 - \dfrac{1}{x-2} - 1$

4.1.3 Yes.

4.2.2 The required equation is $y = \dfrac{1}{2x+2} - 1$.

4.2.3 Observe that f is the function

$$f : \begin{array}{ccc} [-4;4] & \to & [-2;4] \\ x & \mapsto & f(x) \end{array}.$$

Let a be the function with curve $y = 2f(x)$. Then $a : \begin{array}{ccc} [-4;4] & \to & [-4;8] \\ x & \mapsto & a(x) \end{array}.$

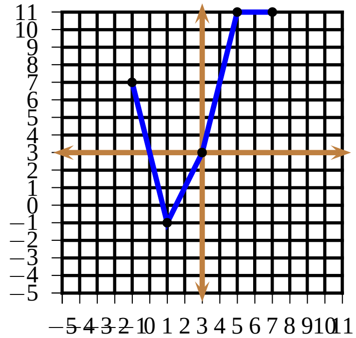


Figure F.13: $y = 2f(x)$.

Let b be the function with curve $y = f(2x)$. Then $b : \begin{array}{ccc} [-2;2] & \to & [-2;4] \\ x & \mapsto & b(x) \end{array}.$

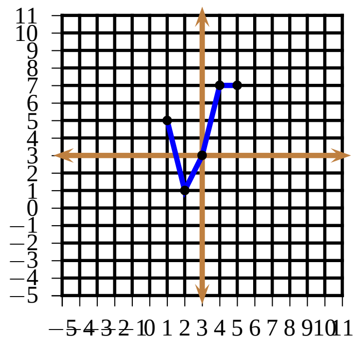


Figure F.14: $y = f(2x)$.

Let c be the function with curve $y = 2f(2x)$. Then $c : \begin{array}{ccc} [-2;2] & \to & [-4;8] \\ x & \mapsto & c(x) \end{array}.$

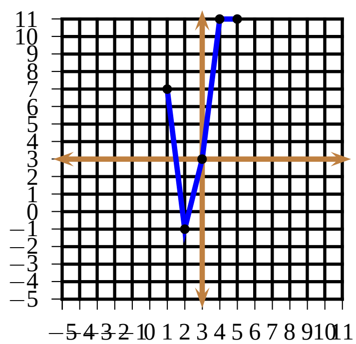


Figure F.15: $y = 2f(2x)$.

4.3.1 Proceeding successively:

1. A reflexion about the x-axis gives the curve
$$y = -f(x) = |x| - 2 = a(x),$$
say.
2. A translation 3 units up gives the curve
$$y = a(x) + 3 = |x| + 1 = b(x),$$
say.
3. A horizontal stretch by a factor of $\frac{3}{4}$ gives the curve
$$y = b\left(\frac{4}{3}x\right) = \left|\frac{4x}{3}\right| + 1 = \frac{4}{3}|x| + 1 = c(x),$$
say. Observe that the resulting curve is
$$y = c(x) = b\left(\frac{4}{3}x\right) = a\left(\frac{4}{3}x\right) + 3 = -f\left(\frac{4}{3}x\right) + 3.$$

4.3.2 (1) $y = -(x+1)(x+2) - 2$ (2) $y = -2x - 7$ (3) $y = |1 - x| - 1$

4.3.3 The graphs are shewn below.

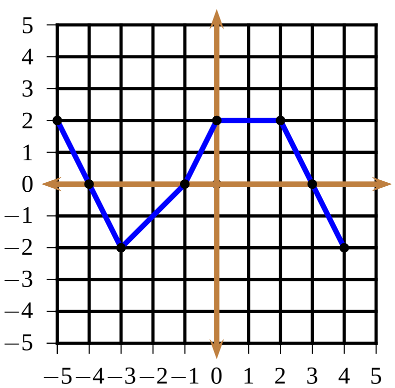


Figure F.16: $y = f(x+1)$.

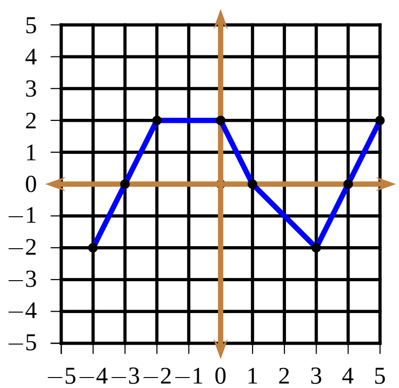


Figure F.17: $y = f(1-x)$.

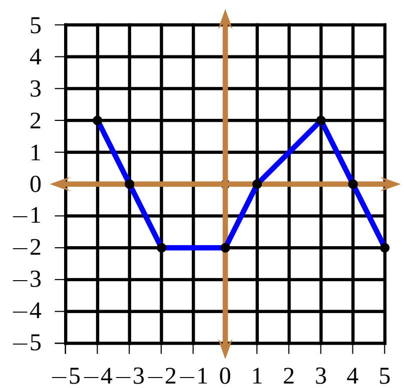


Figure F.18: $y = -f(1-x)$.

4.4.1 Here are the required graphs.

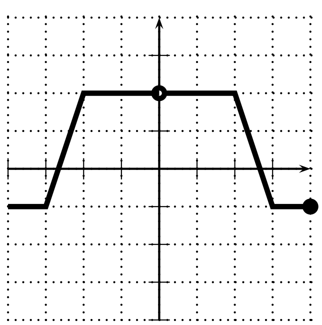


Figure F.19: Even completion.

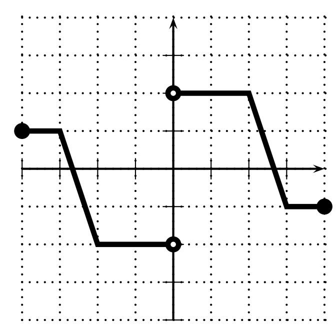


Figure F.20: Odd Completion.

4.4.2 Since f is even, $f(2) = 3$, $f(-3) = 2$. Since g is odd, $g(2) = -2$, $g(-3) = -4$. Thus
$$(f+g)(2) = f(2) + g(2) = 3 + (-2) = 1, \qquad (g \circ f)(2) = g(f(2)) = g(3) = 4.$$

4.4.3 Since f is odd, $f(-0) = -f(0)$. But $f(-0) = f(0)$, giving $f(0) = -f(0)$, that is, $2f(0) = 0$ which implies that $f(0) = 0$.

4.4.4 The constant function $\mathbb{R} \to \{0\}$ with assignment rule $f : x \mapsto 0$ is both even and odd. It is the only such function, for if g were both even and odd and $g(x) = a \neq 0$ for some real number x, then we would have $a = g(x) = g(-x) = -g(x) = -a$, implying that $a = 0$.

4.4.5 We will shew that $A = \{0\}$ and consequently, $B = \{f(0)\}$. Let $x \in A$. If $x \neq 0$ then $-x$ must also be in A because f is even. Thus then $x \neq -x$ and $f(x) = f(-x)$, which means that f in not injective and hence not invertible, a contradiction. This means that the only element of A is $x = 0$. In turn, since f is surjective, B must have exactly one element, which perforce must be $f(0)$.

4.5.1 Here are the graphs of $x \mapsto 2f(x)$, $x \mapsto f(2x)$, $x \mapsto f(-x)$ and $x \mapsto -f(x)$.

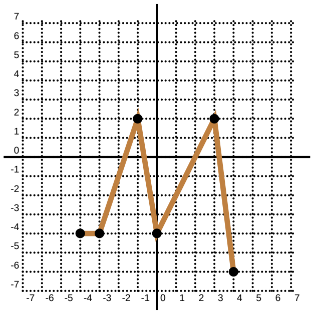


Figure F.21: $y = 2f(x)$

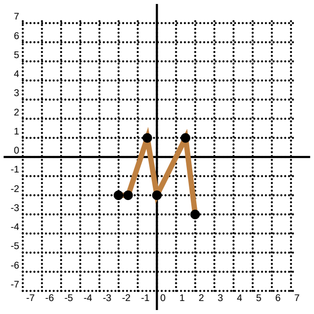


Figure F.22: $y = f(2x)$

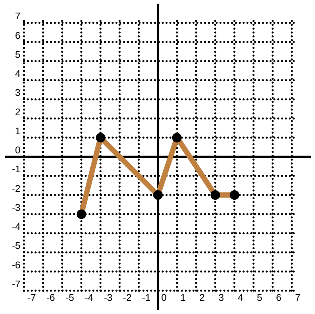


Figure F.23: $y = f(-x)$

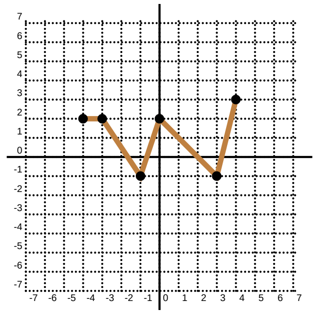


Figure F.24: $y = -f(x)$

Here are the graphs of $x \mapsto -f(-x)$, $x \mapsto f(|x|)$, $x \mapsto |f(x)|$ and $x \mapsto f(-|x|)$.

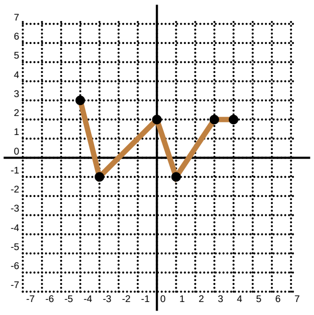


Figure F.25: $y = -f(-x)$

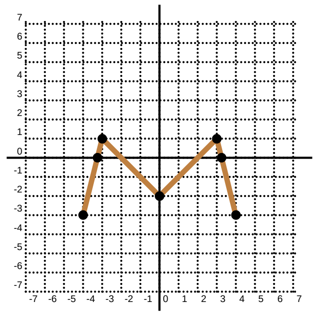


Figure F.26: $y = f(|x|)$

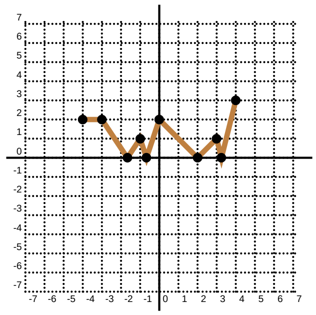


Figure F.27: $y = |f(x)|$

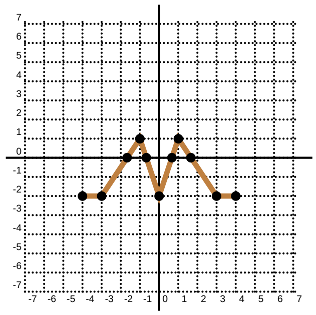


Figure F.28: $y = f(-|x|)$

4.5.2 The graph appears in figure F.29.

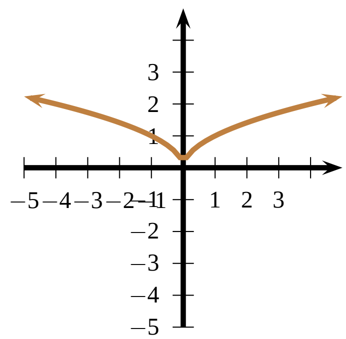


Figure F.29: Problem 4.5.2.

4.5.3 The graphs appear below.

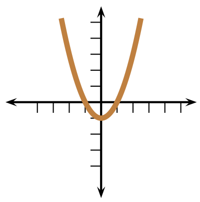


Figure F.30: $y = g(x) = x^2 - 1$

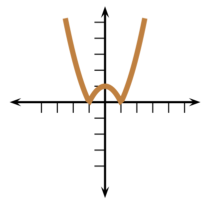


Figure F.31: $y = |g(x)| = |x^2 - 1|$

4.5.4 Observe that $y = \sqrt{x^2 + 2x + 3}$ is an upper semicircle and that

$$y = \sqrt{-x^2 + 2x + 3} \implies x^2 - 2x + y^2 = 3 \implies (x-1)^2 + y^2 = 4,$$

from where the semicircle has radius 2 and centre at $(1, 0)$, as appears in figure F.32.

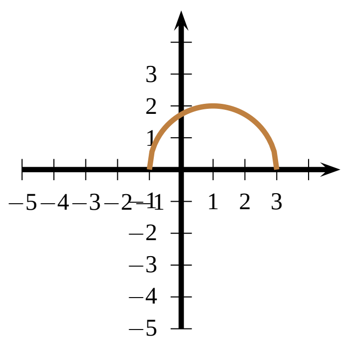


Figure F.32: Problem 4.5.4.

The graph of $y = \sqrt{-x^2 + 2|x| + 3}$ appears in figure F.33. The graph of of $y = \sqrt{-x^2 - 2|x| + 3}$ appears in figure F.34.

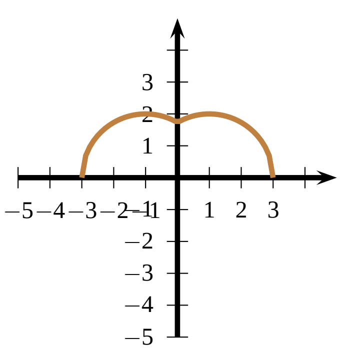


Figure F.33: Problem 4.5.4.

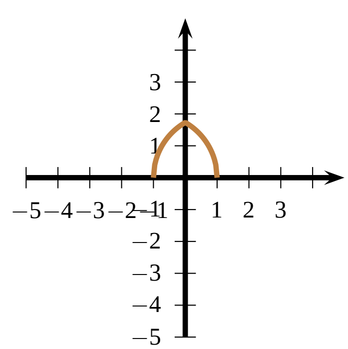


Figure F.34: Problem 4.5.4.

4.5.5 Here is the graph of $y = (x-1)^2 - 2$.

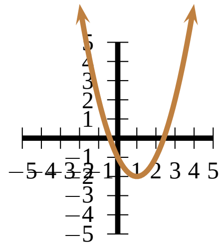


Figure F.35: $y = (x-1)^2 - 2$.

Here is the graph of $y = |(x-1)^2 - 2|$.

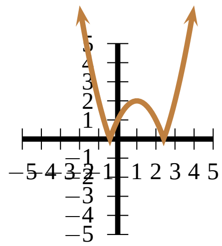


Figure F.36: $y = |(x-1)^2 - 2|$.

Here is the graph of $y = (|x|-1)^2 - 2$.

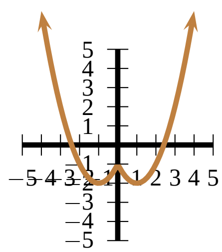


Figure F.37: $y = (|x|-1)^2 - 2$.

Observe that $(-|x|-1)^2 = (-1)^2(|x|+1)^2 = (|x|+1)^2$. Here is the graph of $y = (|x|+1)^2 - 2$.

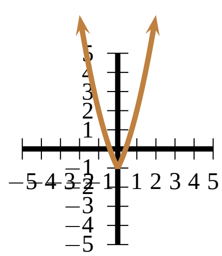


Figure F.38: $y = (|x|+1)^2 - 2$.

4.5.10 Here is the graph of $y = 1 - x$.

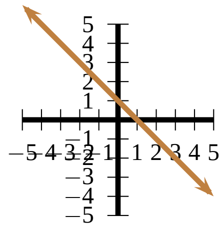


Figure F.39: $y = 1 - x$.

Here is the graph of $y = |1 - x|$.

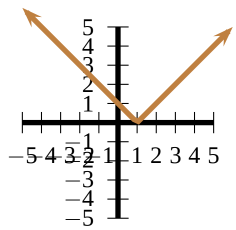


Figure F.40: $y = |1 - x|$.

Here is the graph of $y = 1 - |1 - x|$.

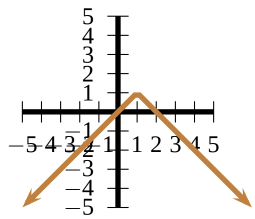


Figure F.41: $y = 1 - |1 - x|$.

Here is the graph of $y = |1 - |1 - x||$.

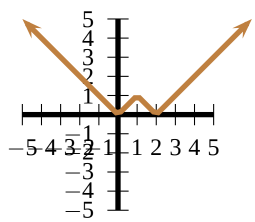


Figure F.42: $y = |1 - |1 - x||$.

Here is the graph of $y = 1 - |1 - |1 - x||$.

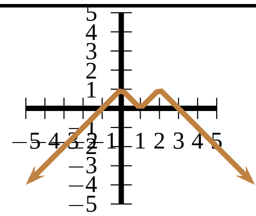


Figure F.43: $y = 1 - |1 - |1 - x||$.

Here is the graph of $y = |1 - |1 - |1 - x|||$.

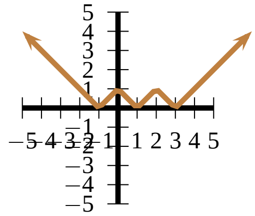


Figure F.44: $y = |1 - |1 - |1 - x|||$.

Here is the graph of $y = 1 - |1 - |1 - |1 - x|||$.

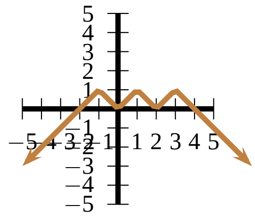


Figure F.45: $y = 1 - |1 - |1 - |1 - x|||$.

Here is the graph of $y = |1 - |1 - |1 - |1 - x||||$.

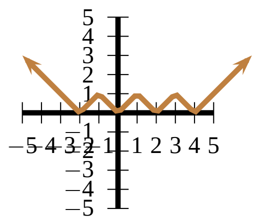


Figure F.46: $y = |1 - |1 - |1 - |1 - x||||$.

4.5.12 Notice that the graph of $y = f(ax)$ is a horizontal shrinking of the graph of $y = f(x)$. Put $g(x) = f(ax)$. Since $g(4/3) = 0$ we must have $4a/3 = -2 \implies a = -3/2$, so the point $(-2,0)$ on the original graph was mapped to the point $(4/3,0)$ on the new graph. Hence the point $(3,0)$ in the old graph gets mapped to $(-2,0)$ and so $C = -2$.

4.6.1 For $x \neq 1$ we have $f(x) = \dfrac{x^2-1}{x-1} = x+1$. Since $f(1-) = 2$ and $f(1+) = 2$ we need $a = f(1) = 2$.

4.6.2 Take, among many possible examples, the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \dfrac{1}{x^3 - x}$ for $x \notin \{-1,0,1\}$ and $f(-1) = f(0) = f(1) = 0$.

4.6.3 We have $f(1-)=0$ and $f(1+)=2+3a$. We need then $0=2+3a$ or $a=-\dfrac{2}{3}$.

4.6.4 For $x\neq 1$ we have $f(x)=\dfrac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+\cdots+x^2+x+1$. Since $f(1-)=n$ and $f(1+)=n$ we need $a=f(1)=n$.

4.6.5 Examine the assignment rule $x\mapsto\lfloor x^3\rfloor$.

4.7.1 Examine the assignment rule $r(x)=\lfloor\dfrac{1}{x}\rfloor, x\neq 0$.

4.7.2 Examine the assignment rule $x\mapsto\lfloor\dfrac{x-1}{2}\rfloor$.

4.7.3 Examine the assignment rule $x\mapsto\lfloor\sqrt{|x|}\rfloor$.

4.7.4 The function is periodic of period 1. If $x\in[0;1]$ then

$$||x||=\min(x,1-x).$$

Its graph appears in figure F.47.

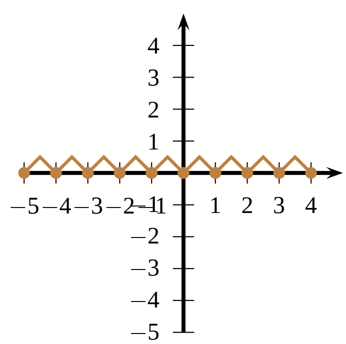


Figure F.47: Problem 4.7.4.

4.7.5 Its graph will jump each time $2x=n$, an integer, that is, when $x=\dfrac{n}{2}$, which means it jumps at every fraction with denominator 2. Its graph appears in figure 4.7.5.

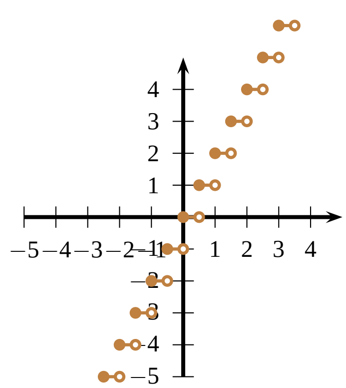


Figure F.48: Problem 4.7.5.

4.7.6 The assertion is false. For example, if $x=2.1$ then $\{2.1\}^2=0.1^2=0.01$ but $\left\{2.1^2\right\}=\{4.41\}=0.41$

4.7.8 The formula $x\mapsto\dfrac{1}{⟦x⟧-\llbracket x\rrbracket}$ is not defined for $x\in\mathbb{Z}$. If $x\in\mathbb{R}\setminus\mathbb{Z}$ then $\dfrac{1}{⟦x⟧-\llbracket x\rrbracket}=1$. Thus the graph consists of the horizontal line of equation $y=1$ but with punctures at the points $(n,1)$, $n\in\mathbb{Z}$.

4.7.9 First consider $n \in \mathbb{Z}$. We have

$$\llbracket x \rrbracket \to \begin{cases} n-1 & \text{as } x \to n- \\ n & \text{as } x \to n+ \end{cases}$$

Then

$$\llbracket x \rrbracket + \sqrt{x - \llbracket x \rrbracket} \to \begin{cases} n-1+\sqrt{n-(n-1)} & = & n & \text{as } x \to n- \\ n+\sqrt{n-n} & = & n & \text{as } x \to n+ \end{cases}$$

We deduce that f is continuous at the integers. Since f is clearly continuous at non-integral points, we conclude that f is everywhere continuous.

5.2.1 To prove that $x \mapsto |x|$ is convex, we use the triangle inequality theorem 62 and the fact that $|\lambda| = \lambda$, $|1-\lambda| = 1-\lambda$ for $\lambda \in [0;1]$. We have

$$\begin{aligned} \mathbf{AbsVal}(\lambda a + (1-\lambda)b) &= |\lambda a + (1-\lambda)b| \\ &\leq |\lambda a| + |(1-\lambda)b| \\ &= \lambda|a| + (1-\lambda)|b| \\ &= \lambda\mathbf{AbsVal}(a) + (1-\lambda)\mathbf{AbsVal}(b), \end{aligned}$$

whence $x \mapsto |x|$ is convex. As $\mathbf{AbsVal}(-x) = |-x| = |x| = \mathbf{AbsVal}(x)$, the absolute value function is an even function. For $a < b < 0$,

$$\frac{\mathbf{AbsVal}(b) - \mathbf{AbsVal}(a)}{b-a} = \frac{|b|-|a|}{b-a} = \frac{-b-(-a)}{b-a} = -1 < 0,$$

$x \mapsto |x|$ is a strictly decreasing function for $x < 0$. Similarly, for $0 < a < b$

$$\frac{\mathbf{AbsVal}(b) - \mathbf{AbsVal}(a)}{b-a} = \frac{|b|-|a|}{b-a} = \frac{b-a}{b-a} = 1 > 0,$$

and so $x \mapsto |x|$ is a strictly increasing function for $x > 0$. Also, assume that $y \in \mathbf{Im}(\mathbf{AbsVal})$. Then $\exists x \in \mathbb{R}$ with $y = \mathbf{AbsVal}(x) = |x|$, which means that $y \geq 0$ and so $\mathbf{Im}(\mathbf{AbsVal}) = [0;+\infty[$.

To obtain the graph of $x \mapsto |x|$ we graph the line $y = -x$ for $x < 0$ and the line $y = x$ for $x \geq 0$.

5.4.2 (i) $y = (x+3)^2$ vertex at $(-3,0)$, (ii) $y = (x+6)^2 - 1$ vertex at $(-6,-1)$, (iii) $y = (x+1)^2 - 16$, vertex at $(-1,-16)$ (iv) $y = -(x-\frac{1}{2})^2 + \frac{1}{4}$, vertex at $(\frac{1}{2},\frac{1}{4})$ (v) $y = 2(x-3)^2+5$, vertex at $(3,5)$, (vi) $3(x-\frac{1}{3})^2 + \frac{5}{9}$, vertex at $(\frac{1}{3},\frac{5}{9})$ (vii) $y = \frac{1}{5}(x+5)^2+8$, vertex at $(-5,8)$

5.4.3 $(3,-9)$

5.4.4 $y = 2x^2 - 1$

5.4.5 $y = -2(x+3)(x-4)$

5.4.6 Observe that $x(1-x) = \dfrac{1}{4} - (x - \dfrac{1}{2})^2 \leq \dfrac{1}{4}$ and that for $x \in [0,1], 0 \leq x(1-x)$. Thus if all these products are $> \frac{1}{4}$ we obtain $\dfrac{1}{4^3} < a(1-b)b(1-c)c(1-a) = a(1-a)b(1-b)c(1-c) \leq \dfrac{1}{4^3}$, a contradiction. Thus one of the products must be $\leq \frac{1}{4}$.

5.4.7 $P(x) = 21025 - 25(x-1)^2$; $\$21025$

5.4.8 We have

$$\begin{aligned} |x^2-2x| = |x^2+1| &\iff (x^2-2x = x^2+1) \text{ or } (x^2+2x = -x^2-1) \\ &\iff (-2x-1=0) \text{ or } (2x^2+2x+1=0) \\ &\iff \left(x = -\frac{1}{2}\right) \text{ or } \left(x = -\frac{1}{2} \pm \frac{i}{2}\right), \end{aligned}$$

whence the solution set is $\left\{-\dfrac{1}{2}\right\}$.

5.4.9 We have

$$\begin{aligned}
(x^2+2x-3)^2 = 2 \quad &\iff \quad (x^2+2x-3=\sqrt{2}) \text{ or } (x^2+2x-3=-\sqrt{2}) \\
&\iff \quad (x^2+2x-3-\sqrt{2}=0) \text{ or } (x^2+2x-3+\sqrt{2}=0) \\
&\iff \quad \left(x = \frac{-2\pm\sqrt{4-4(-3-\sqrt{2})}}{2}\right) \\
&\qquad \text{or } \left(x = \frac{-2\pm\sqrt{4-4(-3+\sqrt{2})}}{2}\right) \\
&\iff \quad \left(x = \frac{-2\pm\sqrt{16+4\sqrt{2}}}{2}\right) \\
&\qquad \text{or } \left(x = \frac{-2\pm\sqrt{16-4\sqrt{2}}}{2}\right) \\
&\iff \quad (x=-1\pm\sqrt{4+\sqrt{2}}) \text{ or } (x=-1\pm\sqrt{4-\sqrt{2}}).
\end{aligned}$$

Since each of $4\pm\sqrt{2}>0$, all four solutions found are real. The set of solutions is $\left\{-1\pm\sqrt{4\pm\sqrt{2}}\right\}$.

5.4.10

$$\begin{aligned}
x^3-x^2-9x+9=0 \quad &\iff \quad x^2(x-1)-9(x-1)=0 \\
&\iff \quad (x-1)(x^2-9)=0 \\
&\iff \quad (x-1)(x-3)(x+3)=0 \\
&\iff \quad x\in\{-3,1,3\}.
\end{aligned}$$

5.4.11

$$\begin{aligned}
x^3-2x^2-11x+12=0 \quad &\iff \quad x^3-x^2-x^2+x-12x+12=0 \\
&\iff \quad x^2(x-1)-x(x-1)-12(x-1)=0 \\
&\iff \quad (x-1)(x^2-x-12)=0 \\
&\iff \quad (x-1)(x+3)(x-4)=0 \\
&\iff \quad x\in\{-3,1,4\}.
\end{aligned}$$

5.4.12 $x^3-1=(x-1)(x^2+x+1)$. If $x\neq 1$, the two solutions to $x^2+x+1=0$ can be obtained using the quadratic formula, getting $x=1/2\pm i\sqrt{3}/2$. There is only one real solution, namely $x=1$.

5.4.13 The parabola has equation of the form $x=a(y-k)^2+h=a(y-2)^2+1$. Since when $x=3$ we get $y=4$, we have,

$$3=a(4-2)^2+1 \implies 3=4a+1 \implies a=\frac{1}{2}.$$

The equation sought is thus

$$x=\frac{1}{2}(y-2)^2+1.$$

5.4.14 Observe that

$$x^{-4}-10x^{-2}+9=(x^{-2}-9)(x^{-2}-1).$$

Thus $\dfrac{1}{x^2}=9$ and $\dfrac{1}{x^2}=1$, whence $x=\pm\frac{1}{3}$ and $x=\pm 1$.

5.4.15 Rearranging,

$$(t^2-81)x=3(t-9) \implies (t-9)(t+9)x=3(t-9). \tag{F.1}$$

If $t=9$, (F.1) becomes $0=0$, which will be true for all values of x. If $t=-9$, (F.1) becomes $0=-54$, which is clearly nonsense. If $t\in\mathbb{R}\setminus\{-9,9\}$, then

$$x=\frac{3}{t+9}$$

is the unique solution to the equation.

5.4.16 Let x and $50-x$ be the numbers. We seek to maximise the product $P(x)=x(50-x)$. But $P(x)=50x-x^2=-(x^2-50x)=-(x^2-50x+625)+625=625-(x-25)^2$. We deduce that $P(x)\leq 625$, as the square of any real number is always positive. The maximum product is thus 625 occurring when $x=25$.

5.4.17 If b,h are the base and height, respectively, of the rectangle, then we have $20=2b+2h$ or $10=b+h$. The area of the rectangle is then $A(h)=bh=h(10-h)=10h-h^2=25-(h-5)^2$. This shows that $A(h)\leq 25$, and equality occurs when $h=5$. In this case $b=10-h=5$. The height is the same as the base, and so the rectangle yielding maximum area is a square.

5.4.18

1. The current production is $25\times 600=15000$ fruits.
2. If x more trees are planted, the production of each tree will be $600-15x$.
3. Let $P(x)$ be the total production after planting x more trees. Then $P(x)=(25+x)(600-15x)=-15x^2+225x+15000$. A good function modelling this problem is

$$P:\begin{array}{ccc}\{x\in\mathbb{N}|x\geq 25\} & \to & \mathbb{N}\\ x & \mapsto & -15x^2+225x+15000\end{array}.$$

   This model assumes that the amount of trees is never fewer than 25.
4. We maximise $P(x)=-15x^2+225x+15000=15000-15(x^2-15x)=15843.75-15(x-7.5)^2$. The production is maximised if either 7 or 8 more trees are added, in which case the production will be $15843.75-15(7-7.5)^2=15840$ fruits.

5.9.1 Such polynomial must have the form $p(x)=a(x+1)(x-2)(x-3)$, and so we must determine a. But $-24=p(1)=a(2)(-1)(-2)=4a$. Hence $a=-6$. We thus find $p(x)=-6(x+1)(x-2)(x-3)$.

5.9.2 There are ten such polynomials. They are $p_1(x)=-2(x-1)^3$, $p_2(x)=-2(x-2)^3$, $p_3(x)=-2(x-3)^3$, $p_4(x)=-2(x-1)(x-2)^2$, $p_5(x)=-2(x-1)^2(x-2)$, $p_6(x)=-2(x-1)(x-3)^2$, $p_7(x)=-2(x-1)^2(x-3)$, $p_8(x)=-2(x-2)(x-3)^2$, $p_9(x)=-2(x-2)^2(x-3)$, $p_{10}(x)=-2(x-1)(x-2)(x-3)$.

5.9.3 This polynomial must have the form $c(x)=a(x-1)(x+3)^2$. Now $10=c(2)=a(2-1)(2+3)^2=25a$, whence $a=\frac{2}{5}$. The required polynomial is thus $c(x)=\frac{2}{5}(x-1)(x+3)^2$.

5.9.4 Put $g(x)=p(x)-x^2$. Observe that g is also a cubic polynomial with leading coefficient 1 and that $g(x)=0$ for $x=1,2,3$. This means that $g(x)=(x-1)(x-2)(x-3)$ and hence $p(x)=(x-1)(x-2)(x-3)+x^2$. This yields $p(4)=(3)(2)(1)+4^2=22$.

5.9.5 The polynomial $g(x)=p(x)-7$ vanishes at the 4 different integer values a,b,c,d. In virtue of the Factor Theorem,

$$g(x)=(x-a)(x-b)(x-c)(x-d)q(x),$$

where $q(x)$ is a polynomial with integral coefficients. Suppose that $p(t)=14$ for some integer t. Then $g(t)=p(t)-7=14-7=7$. It follows that

$$7=g(t)=(t-a)(t-b)(t-c)(t-d)q(t),$$

that is, we have factorised 7 as the product of at least 4 different factors, which is impossible since 7 can be factorised as $7(-1)1$, the product of at most 3 distinct integral factors. From this contradiction we deduce that such an integer t does not exist.

5.9.6 By the Factor Theorem, we must have

$$\begin{aligned}0=t(-4)=(-4)^3-3a(-4)^2+40\\ \iff 0=-24-48a\\ \iff a=-\frac{1}{2}.\end{aligned}$$

5.9.7 Observe that $f(x)(x-1)=x^5-1$ and

$$f(x^5)=x^{20}+x^{15}+x^{10}+x^5+1=(x^{20}-1)+(x^{15}-1)+(x^{10}-1)+(x^5-1)+5.$$

Each of the summands in parentheses is divisible by x^5-1 and, a fortiori, by $f(x)$. The remainder sought is thus 5.

5.9.8 Put $g(x)=p(x)-x$, then $p(6)=16$.

5.9.10 $(x-2)(x+2)(x-3)$

5.9.11 $(x-3)(x+3)(x+5)(3x-2)$

5.9.12 $a=-7, b=-60$

5.9.13 Let $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ with $a_n\neq 0$, $n\geq 1$. Then

$$16p(x^2)=(p(2x))^2 \implies 16(a_nx^{2n}+a_{n-1}x^{2n-2}+\cdots+a_1x^2+a_0)=\left(2^na_nx^n+2^{n-1}a_{n-1}x^{n-1}+\cdots+2a_1x+a_0\right)^2$$

Since the coefficients on both sides of the equality must agree, we must have

$$16a_n=2^{2n}a_n^2 \implies 2^4=2^{2n}a_n$$

since $a_n\neq 0$. As a_n is an integer, we must have the following cases: $n=1, a_n=4$, $n=2, a_n=1$. Clearly we may not have $n\geq 3$. Thus such polynomials are either linear or quadratic. Also, for $x=0$, $16p(0)=(p(0))^2$ and therefore either $p(0)=0$ or $p(0)=16$.
For $n=1$ we seek $p(x)=4x+a$. Solving

$$16(4x^2+a)=(8x+a)^2 \implies a=0,$$

whence $p(x)=4x$.
For $n=2$, let $p(x)=x^2+ax+b$. Solving

$$16(x^4+ax^2+b)=(4x^2+2ax+b)^2 \implies a=0.$$

Since $p(0)=0$ or $p(0)=16$, we must test $p(x)=x^2$ and $p(x)=x^2+16$. It is easy to see that only $p(x)=x^2$ satisfies the desired properties. In conclusion, $4x$ and x^2 are the only two such polynomials..

7.2.1 F; F; F; T; F; T

7.3.2 $\frac{S^5}{3125}$.

7.3.4 (1) $a(x)\leq\frac{27}{256}$ achieved at $x=\frac{1}{4}$,
(2) $b(x)\leq\frac{1}{16}$ achieved at $x=\frac{1}{2}$,
(3) $c(x)\leq\frac{108}{3125}$ achieved at $x=\frac{2}{5}$, (Hint: Consider $\frac{9}{4}c(x)=(\frac{3}{2}x)(\frac{3}{2}x)(1-x)^3$.)

8.1.1 T; T; T; F; T; T; F; F

8.1.2 (1) -5, (2) -5, (3) $-\frac{5}{3}$, (4) $-\frac{1}{2}$, (5) $\frac{3}{5}$, (6) 6, (7) $\frac{52}{15}$, (8) -4 , (9) $-\frac{1}{4}$, (10) 1

8.1.3 (1) $\frac{2}{5}$, (2) $-\frac{5}{2}$, (3) $-\frac{1}{2}$, (4) 2, (5) $\frac{42}{125}$

8.2.1 (1) $\sqrt[4]{3}$, (2) 81, (3) 64 (4) 5, (5) ± 2, (6) $\log_{23}2$, (7) $\log_2 3,\ \log_3 2,\ 0$, (8) $\log_2 7, 1$, (9) 0, (10) $\log_6 2$, (11) $\log_6 2$, (12) $\log_5 4, \log_5 3$, (13) 81, (14) $\sqrt[3]{5}$

8.3.1 1

8.3.2 F; F; T

8.3.3 (1) 14, (2) $\frac{2}{7}$, (3) $\frac{1}{4}$, (4) $\frac{7}{3}$

8.3.4 3^{1000}

8.3.5 10

8.3.6 $\frac{1}{2}$

8.3.7 $\frac{b}{3}$

8.3.8 $-3s^3+10s^2+2s-3$

8.3.9 $\frac{31}{32}$

8.3.10 (1) $N^{-\alpha\beta\gamma/s}$, (2) 0, (3) 1 (4) 2, (5) N, (6) 0, (7) $\frac{1373}{196}$

8.3.11 (1) About 107.37 km (2) 42 times.

8.3.12 2083

8.3.13 $a=4, b=3, c=24$

8.3.14 $\frac{17}{6}$

9.1.1 F; T; F; F; T; T

9.1.2

1. $\frac{3\pi}{5}$, quadrant II ;
2. $\frac{7\pi}{5}$, quadrant III ;
3. $\frac{7\pi}{5}$, quadrant III;
4. $\dfrac{8\pi}{57}$, quadrant I;
5. $\dfrac{9\pi}{8}$, quadrant III;
6. $\dfrac{6\pi}{79}$, quadrant I;
7. $\dfrac{6\pi}{7}$, quadrant II;
8. 1, quadrant I;
9. 2, quadrant II;
10. 3, quadrant II;
11. 4, quadrant III; (xii) 5, quadrant IV;
12. 6, quadrant IV;
13. $100-30\pi$, quadrant IV;
14. $2\pi-3.14$, quadrant III;
15. $2\pi-3.15$, quadrant II

9.1.3 (i) $\frac{3\pi}{20}$, $\frac{7\pi}{20}$, $\frac{11\pi}{20}$, $\frac{3\pi}{4}$, $\frac{19\pi}{20}$; (ii) $-\frac{17\pi}{20}$, $-\frac{13\pi}{20}$, $-\frac{9\pi}{20}$, $-\frac{\pi}{4}$, $\frac{-\pi}{20}$.

9.1.4 Yes; No.

9.2.2 F; F; T; T; F; T; F; F; T; F; T; T; F; T; F; F

9.2.3 $\cos t = 0.6$

9.2.4 $\sin u = \sqrt{.19}$

9.2.5 $\cos t = \dfrac{3\sqrt{2}}{5}$

9.2.6 $\sin u = -\dfrac{\sqrt{3}}{4}$

9.2.7 $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, $\sin\frac{5\pi}{6} = \frac{1}{2}$

9.2.8 $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

9.2.9 $\sin(\frac{31\pi}{6}) = -\frac{1}{2}$ and $\cos(\frac{31\pi}{6}) = -\frac{\sqrt{3}}{2}$

9.2.10 $\sin(\frac{20\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{20\pi}{3}) = -\frac{1}{2}$

9.2.11 $\sin(\frac{17\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\frac{17\pi}{4}) = \frac{\sqrt{2}}{2}$

9.2.12 $\sin(\frac{-15\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\frac{-15\pi}{4}) = \frac{\sqrt{2}}{2}$

9.2.13 $\sin(\frac{202\pi}{3}) = -\frac{\sqrt{3}}{2}$ and $\cos(\frac{202\pi}{3}) = -\frac{1}{2}$

9.2.14 $\sin(\frac{171\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\frac{171\pi}{4}) = -\frac{\sqrt{2}}{2}$

9.2.27 Hint: Use the Arithmetic-Geometric-Mean Inequality $\frac{a+b}{2} \geq \sqrt{ab}$, for non-negative real numbers a, b.

9.3.1 F; F; F; F

9.4.1 F; F; T; T; F; T; T; F; T; F

9.4.2 $\{-\frac{5\pi}{6}, -\frac{\pi}{6}\}$

9.4.3 $\{\frac{\pi}{12} + \frac{n\pi}{3}, n \in \mathbb{Z}\}$

9.4.4 $\{\pm\frac{2\pi}{3} + 2\pi n, 2\pi n, n \in \mathbb{Z}\}$.

9.4.5 $\{(-1)^{n+1}\frac{\pi}{6} + \frac{n\pi}{3}, n \in \mathbb{Z}\}$; $\{(-1)^{n+1}\frac{\pi}{6} + \frac{n\pi}{3}, n = 295, 296, 297, 298, 299, 300\}$

9.4.6 $\{(2n+1)\pi, n \in \mathbb{Z}\}$

9.4.7 $\{\frac{n\pi}{2},\ n \in \mathbb{Z}\}$

9.4.8 $\emptyset$

9.4.9 $\{-\frac{\pi}{6}+n\pi, \frac{2\pi}{3}+n\pi\}$

9.4.10 (1) $\{-\frac{\pi}{3}, \frac{\pi}{3}\}$; (2) $\{-\frac{\pi}{6}, \frac{\pi}{6}\}$; (3) No solutions in this interval; (4) All the solutions belong to this interval $\{\frac{6}{(-1)^n\pi+2n\pi},\ n\in\mathbb{Z}\}$; (5) $\{-\frac{\pi}{2}, \frac{\pi}{2}\}$

9.4.11 $\frac{2\sqrt{2}}{3}$

9.4.12 $\frac{\sqrt{5}}{3}$

9.4.13 $\frac{\sqrt{5}}{3}$

9.4.14 $5-2\pi$; $4\pi-10$

9.5.1 F; F; F; F

9.5.2 $\sin x = -\frac{2}{3}, \cos x = \frac{\sqrt{5}}{3}, \tan x = -\frac{-2\sqrt{5}}{5}$.

9.5.3 $\sin x = -\frac{2\sqrt{5}}{5}, \cos x = -\frac{\sqrt{5}}{5}$

9.5.4 $\cos x = -\sqrt{1-t^4}, \tan x = -\frac{t^2}{\sqrt{1-t^4}}$

9.5.5 $\sin \operatorname{arcsec} x = -\sqrt{1-\frac{1}{x^2}}$

9.5.6 $\frac{3\sqrt{10}}{10}$

9.5.7 $2\pi-6$; $4\pi-10$

9.6.3 $\cos(\pi/12) = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$, $\sin(\pi/12) = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$.

9.6.4 $\frac{\cot a \cot b - 1}{\cot a + \cot b}$

9.6.5 $\frac{1}{2}\cos x - \frac{1}{2}\cos 3x$

9.6.6 $\frac{1}{2}\cos 3x + \frac{1}{2}\cos 5x$

9.6.7 $-\arcsin \frac{4\sqrt{15}-3}{20}$

9.6.8 $\arctan \frac{1}{13}$.

9.6.9 $\pi + \arctan \frac{1}{8}$

9.6.10 $\frac{1}{2}\sin 3x - \frac{1}{2}\sin x$

9.6.11 $\frac{1}{4}\sin 2x + \frac{1}{4}\sin 4x - \frac{1}{4}\sin 6x$

9.6.12 $-(\frac{\sqrt{2}}{6}+\frac{\sqrt{3}}{6})$

9.6.13 $x = \pm\frac{\pi}{4}+n\pi,\ x = \pm\frac{\pi}{2}+2n\pi,\ n\in\mathbb{Z}$

9.6.14 $x=0$.

9.6.15 $x=0$ or $x=1$.

9.6.16 $x = \frac{\sqrt{17}-3}{4}$

A.3.1 Using the binomial theorem and Euler's formula,

$$\begin{aligned}
32\cos^6 2x &= \left(e^{2ix}+e^{-2ix}\right)^6\\
&= \binom{6}{0}e^{12ix}+\binom{6}{1}e^{10ix}e^{-2ix}+\binom{6}{2}e^{8ix}e^{-4ix}+\binom{6}{3}e^{6ix}e^{-6ix}+\binom{6}{4}e^{4ix}e^{-8ix}+\binom{6}{5}e^{2ix}e^{-10ix}+\binom{6}{6}e^{-12ix}\\
&= e^{12ix}+6e^{8ix}+15e^{4ix}+20+15e^{-4ix}+6e^{-8ix}+e^{-12ix}\\
&= (e^{12ix}+e^{-12ix})+6(e^{8ix}+e^{-8ix})+15(e^{4ix}+e^{-4ix})+20\\
&= 2\cos 12x+12\cos 8x+30\cos 4x+20,
\end{aligned}$$

from where we deduce the result.

A.3.2 From

$$\cos 3x = 4\cos^3 x - 3\cos x, \qquad \sin 3x = 3\sin x - 4\sin^3 x,$$

we gather, upon using the double angle and the sum identities,

$$\begin{aligned}
\tan 3x &= \frac{3\sin x-4\sin^3 x}{4\cos^3 x-3\cos x}\\
&= \tan x\left(\frac{3-4\sin^2 x}{4\cos^2 x-3}\right)\\
&= \tan x\left(\frac{3-4\sin^2 x}{1-4\sin^2 x}\right)\\
&= \tan x\left(1+\frac{2}{1-4\sin^2 x}\right)\\
&= \tan x+\frac{2\sin x}{\cos x-4\sin^2 x\cos x}.\\
&= \tan x+\frac{2\sin x}{\cos x-2\sin x\sin 2x}\\
&= \tan x+\frac{2\sin x}{\cos x-2\left(\dfrac{\cos x}{2}-\dfrac{\cos 3x}{2}\right)}\\
&= \tan x+\frac{2\sin x}{\cos 3x}.
\end{aligned}$$

Finally, upon letting $x=\dfrac{\pi}{9}$ we gather,

$$\sqrt{3}=\tan\frac{\pi}{3}=\tan\frac{\pi}{9}+\frac{2\sin\dfrac{\pi}{9}}{\cos\dfrac{\pi}{3}}=\tan\frac{\pi}{9}+4\sin\frac{\pi}{9},$$

as it was to be shewn.

C.1.1 (1) 2, -1, 5, -7, 17; (2) 2, $1/2$, $5/4$, $7/8$, $17/16$; (3) 2, 2, 3, 7, 25; (4) $1/3$, $1/5$, $1/25$, $1/119$, $1/721$; (5) 2, $9/4$, $64/27$, $625/256$, $7776/3125$

C.1.2 (1) Strictly increasing, unbounded (2) non-monotonic, unbounded (3) strictly decreasing, bounded (4) strictly increasing, bounded (5) strictly increasing, unbounded, (6) non-monotonic, bounded, (7) strictly increasing, bounded, (8) strictly decreasing, bounded

C.3.1 $-\dfrac{2}{3}$

C.3.2 One is given that $ar^5=20$ and $ar^9=320$. Hence $|ar^2|=\frac{5}{2}$

C.3.3 (1) $\frac{3^{50}-1}{2}=358948993845926294385124$, (2) $\frac{1-y^{101}}{1-y}$, (3) $\frac{1+y^{101}}{1+y}$, (4) $\frac{1-y^{102}}{1-y^2}$

C.3.4 At 2 : 00 : 59 PM (the second just before 2 : 01 PM.)

C.3.5 2^{30}

C.3.6 (1) $2^{63}=9223372036854775808$, (2) $2^{64}-1=18446744073709551614$, (3) 1.2×10^{15} kg, or 1200 billion tonnes (4) 3500 years

C.4.1 (1) $\frac{64}{15}$, (2) $\frac{27}{29}$, (3) $\frac{140+99\sqrt{2}}{8}$, (4) $\frac{27\sqrt{6}+18\sqrt{2}}{46}$, (5) $\frac{3+\sqrt{5}}{2}$, (6) diverges, (7) $\frac{1}{1+x}$, (8) $\frac{3}{2}$, (9) $\frac{x^2}{x-y}$

C.4.2 (1) $\frac{1}{3}$, (2) $\frac{2}{3}$, (3) $\frac{23}{90}$, (4) $\frac{21023}{9900}$, (5) $\frac{3}{7}$

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